WARSAW UNIVERSITY
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Ph.D. THESIS

Michał Macias, M.Sc.

Fractional Variable-Order Models of Particular Dynamical Systems

Supervisor
Professor Dominik Sierociuk

Co-supervisor
Ph.D. Wiktor Malesza

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To my Family

Abstract

This thesis is addressed to analog modeling of dynamical systems using fractional variable-order calculus. Throughout this thesis, the dynamical systems, which can be effectively modeled by fractional variable-order calculus, will be called the fractional variable-order (dynamical) systems.

At the beginning, the particular types of fractional variable-order operators and their equivalent switching order structures were introduced. It is worth to notice, that such switching schemes can be used as interpretations of variable-order operators. Then, the switching order structures corresponding to $\mathcal{A}$,$\mathcal{B}$,$\mathcal{D}$,$\mathcal{E}$-type operators were applied to design the dynamical systems. Further, based on variable-order analog models, the experimental data were collected and compared to their numerical counterparts.

In order to create the analog models of variable-order systems, the fractional order impedances were built as a domino ladder structure. The initial value of voltage occurring at the impedance's branches, can cause inaccuracies during experimental setup. To deal with it, the method concerning the initial conditions in the recursive Grünwald-Letnikov, $\mathcal{D}$ and $\mathcal{E}$-type definitions, in the form of finite and infinite length, were proposed and experimentally validated.

It was also shown, that presented in the thesis variable-order operators posses the duality property. The duality property is understood as an appropriate composition of two particular fractional variable-order operators, which for opposite value of orders gives an original function. Moreover, the composition properties for the $\mathcal{E}$-type operator with fractional constant-order Grünwald-Letnikov definition from left- and right-hand side were proved.

Keywords: fractional calculus, fractional order derivative, fractional variable-order derivative, analog modeling

Streszczenie

Niniejsza rozprawa traktuje o modelowaniu analogowym wybranych układów dynamicznych z użyciem pochodno-całek niecałkowitego zmiennego rzędu. W pracy przedstawiono wybrane definicje pochodno-całek niecałkowitego zmiennego rzędu oraz równoważne im schematy przełączeń. Warto podkreślić, że przytoczone schematy przełączeń prezentują zachowanie oraz mogą stanowić interpretację wybranych operatorów. Następnie, schematy przełączeń odpowiadające poszczególnym definicjom ($\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$, $\mathcal{E}$) pochodno-całek niecałowitego zmiennego rzędu zostały wykorzystane do realizacji modeli analogowych wybranych układów dynamicznych. Dokładność zrealizowanych modeli analogowych została poddana weryfikacji eksperymentalnej, a otrzymane wyniki porównane z odpowiadającymi im wynikami numerycznymi.

Dla układów całkujących niecałkowitego stałego, a w ogólnym przypadku zmiennego rzędu, których modele analogowe oparte są na drabinkowej realizacji impedancji niecałkowitego rzędu, bardzo istotne okazuje się napięcie początkowe zgromadzone w jej poszczególnych gałęziach. Różne wartości tego napięcia potrafią znacząco wpłynąć na wyniki eksperymentalne. Uwzględniając ten fakt, dla rekurencyjnych postaci operatorów: Grünwalda-Letnikova, definicji typu $\mathcal{D}$ oraz definicji typu $\mathcal{E}$, pokazano metody uwzględniające podane warunki początkowe w układzie. Rozważania teoretyczne związane z warunkami początkowymi zostały także zweryfikowane w badaniach eksperymentalnych.

Mając na uwadze, że definicje niecałkowitego zmiennego rzędu nie spełniają prawa składania operatorów, dla wybranego typu definicji przeanalizowano prawo składania operatora z definicją niecałkowitego stałego rzędu w postaci Grünwalda-Letnikova.
Udowodniono, że lewostronne oraz prawostronne składanie tego operatora z definicją stałego rzędu nie są tożsame, co prowadzi do znaczących różnic. Jednocześnie przytoczono istotną własność prezentowanych operatorów niecałkowitego zmiennego rzędu nazywaną dualnością. Własność ta powoduje, że złożenie dwóch odpowiednich (różnych) operatorów o przeciwnych rzędach daje w efekcie funckję wejściową.

Słowa kluczowe: rachunek różniczkowy niecałkowitego rzędu, pochodna niecałkowitego rzędu, pochodna niecałkowitego zmiennego rzędu, modelowanie analogowe

Preface

The fractional calculus is understood as a generalization of classical, integer order differentiation and integration onto arbitrary order operators. It is a theory of integrals and derivatives with real and even complex orders. This idea owes its origin to great mathematicians: Gottfried Wilhelm Leibniz and marquis de l'Hôpital, when they considered the notation of half-order derivative in 1695. Two years later Leibniz in the letter to Bernoulli presented the first possible definition of fractional order derivative. The overall concepts of fractional calculus were published in [1, 13, 15, 16, 19, 21, 22, 24, 39, 42, 55, 62, 66].

The revealing of special properties of fractional calculus in late 20th Century, caused the engineers' attention on this area. Engineers realized that, the fractional calculus can be used as a convenience tool to describe and model the real applications in more accurate way. Especially, the fractional calculus was successfully adapted in the area of diffusion processes, where for example fractional order models were used to characterized the behavior of ultracapacitors and fuel cells [12, 14, 94, 110]. Moreover, based on fractional order derivative the heat transfer process is well described in [4, 23, 81, 95, 108, 109] and in many others papers. Fractional calculus was also recognized in mechanical systems modeling, e.g., results for electrical drive system with flexible shaft were presented in [30].

On the other hand, the fractional description of the plant forced the various approach in control theory and its analysis. Thus, the plant could be modeled by fractional calculus and controlled by fractional orders controllers [2, 9, 11, 28, 39, 45, 53, 61, 96, 100]. The practical realizations of fractional order PID controllers, where integral and derivative actions can take non-integer orders, are investigated in [8, 10, 57].

Simultaneously, to control methods the advanced algorithms, necessary to make efficient implementations and identifications of fractional order systems were developed [3, 5, 38, 41, 43, 44, 47, 83, 97]. Moreover, the numerical schemes for solving ordinary and partial fractional order differential equations, based on matrix approach, were presented in [46, 54, 56, 58].

A key issue of each fractional order system is highlighted onto interpretation of its initial conditions. Depending on definition, there exists a long memory effect, which leads to many problems in real plant applications. In [25, 27] the distinction between two primary in fractional calculus definitions–Riemann-Liouville and Caputo were investigated. In [63] an interpretation of initial conditions based on internal state of the frequency distributed fractional integrator model was given. Article [17] presents physical interpretation of initial conditions for different practical examples. The initialization issue of linear fractional differential equations was deeply studied in [101].

Nowadays, a time-varying fractional order systems are intensively studied in a wide range of areas. The variable-order differ-integrals are especially applied in branches with varying nature of process. In [67], the fractional variable-order systems with temperature influence to the order function were suggested. In [60], authors proposed a variable-order differential equation of motion for a spherical particle sedimenting in a quiescent viscous liquid. The particular type of fractional variable-order operators were introduced and successfully applied to anomalous diffusion process in [98, 107]. The fractional variable-order equations are also popular in image processing. For example, in [59] the improvement algorithm of the texture enhancement, which plays an important role in digital and medical image processing, based on variable-order difference, were proposed. In [99], the fractional variable-order differ-integrals were used to described a groundwater flow and environmental remediation. Moreover, the method of analyze a fractional order noise is presented in [68]. Other applications of dealing with variable-order integrals and derivatives in control can be found in [48, 49, 50]. As long as we are spoken about fractional variable-order systems we have to think about their numerical computation as well. In [79, 80, 102, 103] the fractional variable-order tools giving the possibilities to modeling the fractional constant and variable-order systems were provided.

However, in [26, 104], three general types of variable-order derivative definitions were proposed. These definitions were given as a set of mathematics equations without clear interpretation or explanation of their structures. Another papers [67, 102, 105, 106] present additional numerical realizations of fractional variable-order integrators and differentiators. The dynamical properties of variable-order inertial and oscillation system were presented in [40, 51].

Despite of many variable-order definitions, there is a confusion when particular type of definition should be applied. It directly conveys a problem onto control systems designing, because it is hard to implement an efficient control algorithm, with no structure knowledge concerning the behavior of variable-order definition. This was my main motivation to work on it and during studies the following papers [29, 31, 32, 33, 70, 71, 72, 73, 74, 75, 76, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94] try to give an overview on fractional variable-order derivatives as switching order structures.

The main message of the thesis is the following:

"The particular types of fractional variable-order operators are equivalent to switching order structures, which can be used to analog modeling of fractional variable-order dynamical systems."

This thesis is devoted to particular types of fractional variable-order derivatives and their switching structures. Based on it, the fractional variable-order analog models corresponding to appropriate switching schemes are designed and built. Then, the numerical implementations of particular variable-order operators are compared to the experimental data of analog models. The main chapters of the thesis consist of two parts. The first one, presents the theoretical background of particular variable-order derivative. And, in the second part, the theoretical considerations are experimentally validated.

In Chapter 1, the fundamental definitions of fractional constant-order differ-integrals are presented. Additionally, the practical realizations of fractional order impedances used in analog modeling of fractional order integrators are discussed.

Chapter 2 concerns the well-known Grünwald-Letnikov definition and its recursive form. As a novelty, an identity between these two matrix forms is proved. Also, a method for including initial conditions into recursive Grünwald-Letnikov operator is shown and experimentally validated.

Chapter 3 contains the theoretical background of the most known and used $\mathcal{A}$-type fractional variable-order definition. Further, a multi-switching order scheme corresponding to the $\mathcal{A}$-type definition is introduced. Moreover, based on it, the analog model of the $\mathcal{A}$-type integrator is used in purpose to compare the experimental data and their numerical implementations.

In Chapter 4, the $\mathcal{B}$-type fractional variable-order definition is remind. Using the switching order scheme corresponding to such definition, the analog models of the $\mathcal{B}$-type integral and inertial systems are realized and validated in experimental setup.

The two structures of analog models equivalent to the $\mathcal{D}$-type definition are shown in Chapter 5. It is worth to notice, that despite of not too much complicated structure, the multi-swiching analog model allows to change the order in any desire time. Also, a duality property between $\mathcal{D}$ and $\mathcal{A}$–type definitions is discussed. The duality property is a composition of two variable-order definitions, which for opposite value of orders gives an original function. Next, considering the structure of half and quarter–order impedances, the method for including initial conditions in the form of finite and infinite length is given. Finally, both analog models are used to gather the experimental data for zero and non-zero initial conditions and compare them with numerical results.

The most extensive Chapter 6 presents the recursive $\mathcal{E}$-type operator with zero and non-zero initial conditions in the form of finite and infinite length. Further, some properties of the $\mathcal{E}$-type operator such as duality and composition with fractional constant-order derivative based on its switching order scheme are given. The equivalence of the $\mathcal{E}$-type definition with corresponding switching scheme is proved and examined during plenty of experiments. Deeply investigation of analog model is done by realization of the variable-order integral and inertial systems. All achievements and unsolved problems are summarized and discussed in the last chapter.

Claims of Originality

The following novel contributions are made in this dissertation:

- Providing the switching order structure equivalent to the $\mathcal{E}$-type factional variable-order operator.
- Providing the method for involving the initial conditions into $\mathcal{E}$-type fractional variable-order operator in the form of finite and infinite length.
- Providing the order composition properties of the $\mathcal{E}$-type fractional variable-order operator with fractional constant order differ-integral from the left- and right-hand side.
- Analog realization and experimental validation of the $\mathcal{E}$-type fractional variable-order integral and inertial systems.
- Experimental validation of the $\mathcal{E}$-type fractional variable-order systems with initial conditions in the form of infinite length.
- Analog realization and experimental validation of the $\mathcal{B}$-type fractional variable-order integral and inertial systems.
- Analog realization and experimental validation of the $\mathcal{D}$-type integrator based on its series switching order structure.
- Analog realization and experimental validation of the multi-switching (equivalent to $\mathcal{D}$-type) fractional variable-order integral and inertial systems.
- Proving the equivalence between classical and recursive Grünwald-Letnikov operators based on their matrix forms.
- Analog realization and experimental validation of the $\mathcal{A}$-type fractional variable-order integrator.

Contents

Notation

$\mathbb{R}$	the set of real numbers,
$\mathbb{R}^n$	the n-fold cartesian product of $\mathbb{R}$,
$\mathbb{R}^{n\times m}$	the set of $n \times m$ matrices with real entries,
$\mathbb{N}$	the set of integer positive numbers,
$\mathbb{C}$	the set of complex numbers,
$\Gamma(x)$	the Gamma function,
$E_{\alpha,1}$	the one-parameter representation of the Mittag-Leffler function,
$E_{\alpha,\beta}$	the two-parameter representation of the Mittag-Leffler function,
${}^{RL}_{a}I^{\alpha}_{t}f(t)$	the Riemann-Liouville fractional α order integral of function $f(t)$ in the interval $[a\ t]$,
${}^{RL}_{a}\mathrm{D}^{\alpha}_{t}f(t)$	the Riemann-Liouville fractional α order differ-integral of function $f(t)$ in the interval $[a\ t]$,
${}^{C}_{a}\mathrm{D}^{\alpha}_{t}f(t)$	the Caputo fractional α order differ-integral of function $f(t)$ in the interval $[a\ t]$,
$1(t)$	the Heaviside step function,
$I_{k,k}$	the identity square matrix of size k,
s	a second – unit of time,
h	a step time,
V	Volt – unit of voltage,
$\lfloor a \rfloor$	the floor of a (the greatest integer less than or equal to a),
$\Delta^{n}_{f}(z)$	the Z-transform of the signal difference of order α of variable f(t),
${}_{0}\mathrm{D}^{\alpha}_{t}f(t)\quad({}_{0}\Delta^{\alpha}_{l}f_l)$	the Grünwald-Letnikov fractional constant-order α derivative (difference) of $f(t)$ (f_l) function in the interval $[0\ t]$ $([0\ l])$,
${}^{\mathcal{A}}_{0}\mathrm{D}^{\alpha(t)}_{t}f(t)\quad({}^{\mathcal{A}}_{0}\Delta^{\alpha_l}_{l}f_l)$	the $\mathcal{A}$-type fractional variable-order derivative (difference) of $f(t)$ (f_l) function in the interval $[0\ t]$ $([0\ l])$,
${}^{\mathcal{D}}_{0}\mathrm{D}^{\alpha(t)}_{t}f(t)\quad({}^{\mathcal{B}}_{0}\Delta^{\alpha_l}_{l}f_l)$	the $\mathcal{B}$-type fractional variable-order derivative (difference) of $f(t)$ (f_l) function in the interval $[0\ t]$ $([0\ l])$,
${}^{\mathcal{D}}_{0}\mathrm{D}^{\alpha(t)}_{t}f(t)\quad({}^{\mathcal{D}}_{0}\Delta^{\alpha_l}_{l}f_l)$	the $\mathcal{D}$-type fractional variable-order derivative (difference) of $f(t)$ (f_l) function in the interval $[0\ t]$ $([0\ l])$,
${}^{\mathcal{E}}_{0}\mathrm{D}^{\alpha(t)}_{t}f(t)\quad({}^{\mathcal{E}}_{0}\Delta^{\alpha_l}_{l}f_l)$	the $\mathcal{E}$-type fractional variable-order derivative (difference) of $f(t)$ (f_l) function in the interval $[0\ t]$ $([0\ l])$,
${}^{\text{o-s}}_{0}\mathrm{D}^{\alpha(t)}_{t}f(t)\quad({}^{\text{o-s}}_{0}\Delta^{\alpha_l}_{l}f_l)$	the output of the output-switching scheme,
${}^{\text{r-s}}_{0}\mathrm{D}^{\alpha(t)}_{t}f(t)\quad({}^{\text{r-s}}_{0}\Delta^{\alpha_l}_{l}f_l)$	the output of the reductive-switching scheme,
${}^{\text{i-a}}_{0}\mathrm{D}^{\alpha(t)}_{t}f(t)\quad({}^{\text{i-a}}_{0}\Delta^{\alpha_l}_{l}f_l)$	the output of the input-additive switching scheme,
${}^{\text{o-a}}_{0}\mathrm{D}^{\alpha(t)}_{t}f(t)\quad({}^{\text{o-a}}_{0}\Delta^{\alpha_l}_{l}f_l)$	the output of the output-additive switching scheme,

Chapter 1

Introduction to Fractional Calculus

The primary issue of this chapter is to give an overview onto well-known fractional constant-order differ-integrals and to introduce the practical realization of half- and quarter-order impedances. At the beginning, the important Gamma and Mittag-Leffler functions are shown. Next, the differ-integrals according to Riemann-Liouvile and Caputo definitions are presented together with their transfer functions. At the end of this chapter, the realization of half- and quarter-order impedances are discussed.

1.1 The $\Gamma(x)$ function

The $\Gamma(x)$ function plays a significant role in fractional calculus and it is a factorial generalization onto real or complex arguments. The gamma function can be defined in the Euler (integral) or limit form [55]:

Definition 1.1. The $\Gamma(x)$ function in the Euler form

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \tag{1.1}$$

where $\mathrm{Re}(x) > 0$.

Definition 1.2. The $\Gamma(x)$ function in the limit form

$$\Gamma(x) = \lim_{n\to\infty} \frac{n! n^x}{x(x+1)\cdots(x+n)} \tag{1.2}$$

where $x \in \mathbb{C}$.

The $\Gamma(x)$ function posseses a very useful recursive property:

$$\Gamma(x+1) = x\Gamma(x).$$

As can be seen in Fig. 1.1, for negative values of x it is a discontinuous function.

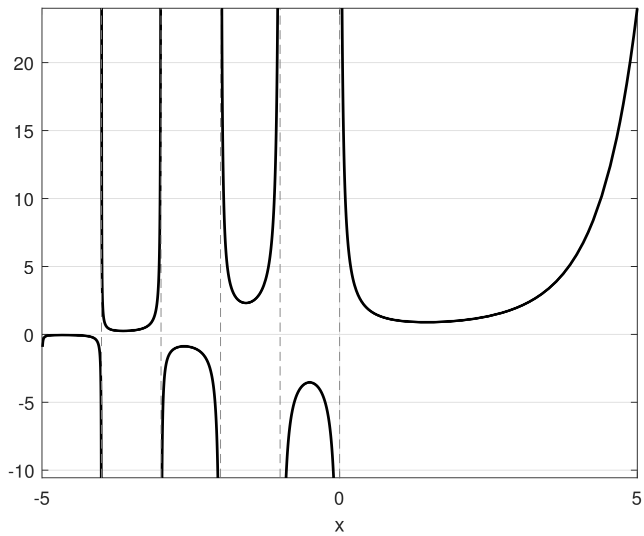


Fig. 1.1. The plot of $\Gamma(x)$ function.

1.2 The Mittag-Leffler function

The Mittag-Leffler function is a generalizaion of exponential function for fractional order differ-integral equations [36, 37].

Definition 1.3. The two-parameters representation of the Mittag-Leffler function is

$$E_{\alpha,\beta} = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)},$$

where $\alpha > 0$ and $\beta > 0$.

If $\beta = 1$, the one-parameter representation of the Mittag-Leffler function is obtained.

Definition 1.4. The one-parameter representation of the Mittag-Leffler function

$$E_{\alpha,\beta} = E_{\alpha,1} = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}.$$

1.3 The Riemann-Liouville differ-integral

To achieve the Riemann-Liouville operator let us begin with Cauchy formula for n-fold integration [18]

$$_aI_t^n f(t) = \int_a^t du_1 \int_a^{u1} du_2 \cdots \int_a^{u_{n-1}} du_n, \tag{1.3}$$

where $n \in \mathbb{N}$, and (a, t) are terminals of the integration.

The more concise form of (1.3) can be written as

$$ {}_aI_t^n f(t) = \frac{1}{(n-1)!}\int_a^t (t-\tau)^{n-1} f(\tau)d\tau. $$

Using the $\Gamma(n)$ function property for factorial generalization, the following Riemann-Liouville fractional order ($\alpha > 0$) integral can be defined

$$ {}_a^{RL}I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \quad \alpha > 0. \tag{1.4} $$

As was shown in [55], (1.4) can be extended to differ-integral definition. Thus we have:

Definition 1.5. The Riemann-Liouville fractional order differ-integral

$$ {}_a^{RL}\mathrm{D}_t^\alpha f(t) = \frac{d^k}{dt^k}{}_aI_t^{k-\alpha} f(t) = \frac{1}{\Gamma(k-\alpha)}\frac{d^k}{dt^k}\int_a^t (t-\tau)^{k-\alpha-1} f(\tau)d\tau, \tag{1.5} $$

where $\alpha \in < k-1; k >$ and $k \in \mathbb{N}$.

Following by [55] the Laplace transform of Riemann-Liouville differ-integral can be expressed by

$$ \mathcal{L}[{}_0^{RL}\mathrm{D}_t^\alpha f(t)] = \begin{cases} s^\alpha F(s) & \text{for } \alpha < 0 \\ s^\alpha F(s) - \sum_{k=0}^{j-1} s^k {}_0^{RL}\mathrm{D}_t^{\alpha-k-1} f(0) & \text{for } \alpha > 0, \end{cases} \tag{1.6} $$

where $j \in \mathbb{N}$ and $\alpha \in < j-1; j >$.

Due to unknown physically interpretation of the fractional order initial conditions occurring in Laplace transform of Riemann-Liouville operator, it carries out difficulties in real applications.

Some addition properties of Riemann-Liouville operator can be found in [35, 39, 42, 55].

1.4 The Caputo differ-integral

To deal with initial conditions of fractional order differ-integral in real applications, the Caputo differ-integral definition was proposed [6, 7]:

Definition 1.6. The Caputo fractional order differ-integral

$$ {}_a^{C}\mathrm{D}_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \tag{1.7} $$

where $\alpha \in (n-1; n >$.

The Laplace transform of Caputo differ-integral can be defined as [55]

$$\mathcal{L}[{}_0^C\mathrm{D}_t^{\alpha} f(t)] = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \tag{1.8}$$

where $\alpha \in (n-1; n)$. More properties of Caputo differ-integral can be found in [35, 42, 55].

1.5 Transfer function of fractional order systems

The transfer function of fractional order systems, under zero initial condition, can be presented as:

Definition 1.7. [55] Transfer function of fractional order systems

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \cdots b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} \cdots a_0 s^{\alpha_0}}, \tag{1.9}$$

where $Y(s)$ is a Laplace transform of output signal, $U(s)$ is a Laplace transform of input signal. The coefficients a_i for $0 \leq i \leq n$ and b_j for $0 \leq j \leq m$ denote system parameters. The orders $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$.

1.6 The fractional constant-order integrator

The fractional constant-order integrator can be described as a transfer function given by [20]

$$G(s) = \frac{k}{s^{\alpha}}. \tag{1.10}$$

Above transfer function can be expressed by the following spectral function

$$G(j\omega) = \frac{k}{(j\omega)^{\alpha}} = \frac{k}{\omega^{\alpha}} e^{-j\alpha\frac{\pi}{2}}. \tag{1.11}$$

Directly from (1.11), the phase and magnitude properties can be obtained, respectively

$$\phi(\omega) = \arg\{G(j\omega)\} = -\alpha\frac{\pi}{2},$$

$$A(\omega) = \frac{k}{\omega^{\alpha}},$$

which yields the following logarithm of magnitude

$$M(\omega) = 20\log(k) - \alpha 20\log(\omega).$$

The magnitude and phase diagrams for selected fractional order integrators are given in Fig. 1.2.

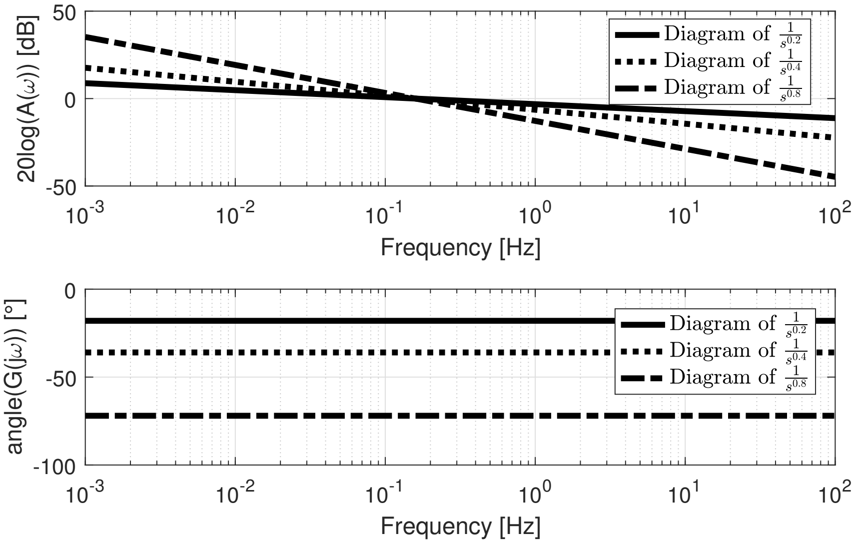


Fig. 1.2. Frequency responses of fractional constant-order integrators for orders $\alpha = \{0.2, 0.4, 0.8\}$.

1.6.1 Analog realization of fractional order impedances

The constant phase $-\alpha\frac{\pi}{2}$ and a magnitude slope equals to $-\alpha 20$ db/dek of fractional constant-order integrator exerted a major impact onto realization of fractional order impedances. The researchers tried to adapt the structures of electrical circuits to fulfill the phase and magnitude criteria. The key issue during the designing of fractional order impedances is to achieve as wide as possible frequency range, where such impedances posses the special features of fractional order behavior. The big influence onto realization has also a structure and value of passive elements used in electrical circuits. Some particular methods devoted to realization of fractional order impedances are shown in [10, 57].

The algorithm of creating the half and quarter impedances introduced first in [69], and meticulously investigated in [52, 77] posses big advantages against to others. It has a simple domino-ladder structure and easily available on the market passive elements of desire values. The real half- and quarter-order impedances based on such domino-ladder structures are used during experimental setup presented in further chapters.

Analog realization of half-order impedance

The scheme of half-order impedance from the method published in [69] is presented in Fig. 1.3. Based on the algorithm, the experimental circuit boards of half-order impedances $Z_{0.5}$ are constructed in the form of domino-ladder structure (see Fig. 1.4). The real circuit

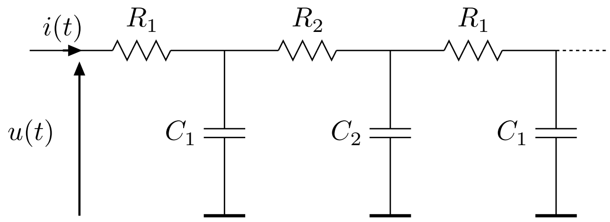


Fig. 1.3. A domino-ladder structure of 0.5 order impedance.

board consists of 200 passive elements with the following values: $R_1 = 2.4\text{k}\Omega$, $R_2 = 8.2\text{k}\Omega$, $C_1 = 330\text{nF}$, and $C_2 = 220\text{nF}$.

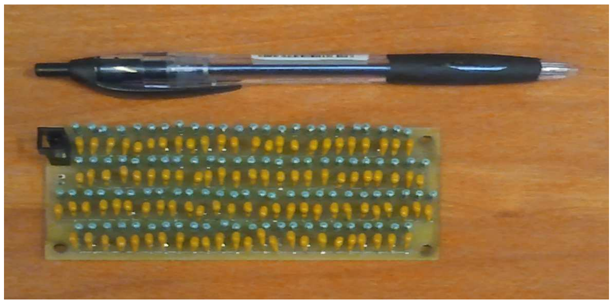


Fig. 1.4. A circuit board of 0.5 order impedance.

Analog realization of quarter-order impedance

The half-order impedance can be extended to build a quarter-order impedance. This can be done by replacing the capacitors in the scheme shown in Fig. 1.3 with half-order impedances. This manipulation gives a $\alpha = 0.25$ order impedance depicted in Fig. 1.5.

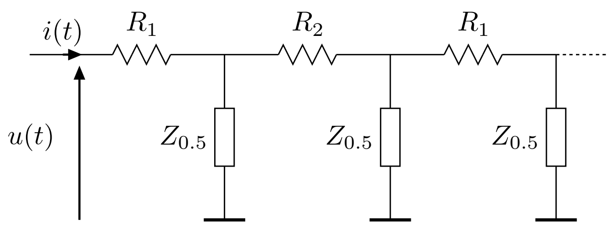


Fig. 1.5. An analog model of 0.25 order impedance.

The real circuit board of quarter-order impedance contains 5000 passive elements and is designed according to Fig. 1.5. The main ladder includes 50 branches with the following resistors' values: $R_1 = 2.4\text{k}\Omega$, $R_2 = 8.2\text{k}\Omega$. The analog model of quarter-order impedance is shown in Fig. 1.6

Experimental validation of fractional order impedances

The validation of fractional order impedances was done based on fractional constant-order integrator. Considering the Z impedance depicted in Fig. 1.7 as a real circuit

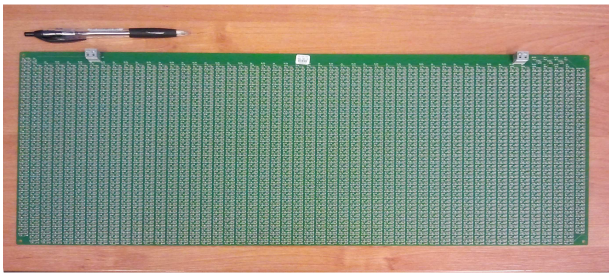


Fig. 1.6. A circuit board of 0.25 order impedance.

boards representing half- or quarter-order impedance, the 0.5 or 0.25 order integrators are achieved. The resistors $R = 14\text{k}\Omega$ for both half- and quarter-order integrators. Due to the inverter signal polarization of the single operational amplifier, the proportional gain -1 based on amplifier A_2 is added in series to amplifier A_1. The experimental data collected

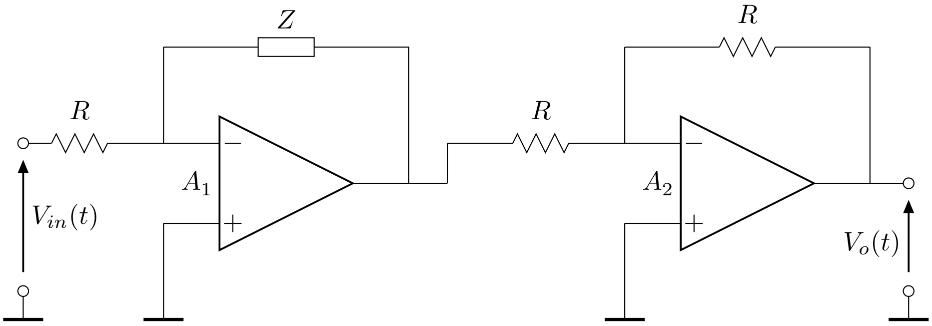


Fig. 1.7. An analog model of fractional constant-order integrator.

to depict the frequency responses of 0.5 and 0.25 order integrators can be observed in Fig. 1.8. It is well visible that analog models fulfill the phase and magnitude properties in wide range of frequencies. However, they still have some frequency constraints, which are also reflect in the time-domain. Comparing the step response of half-order integrator

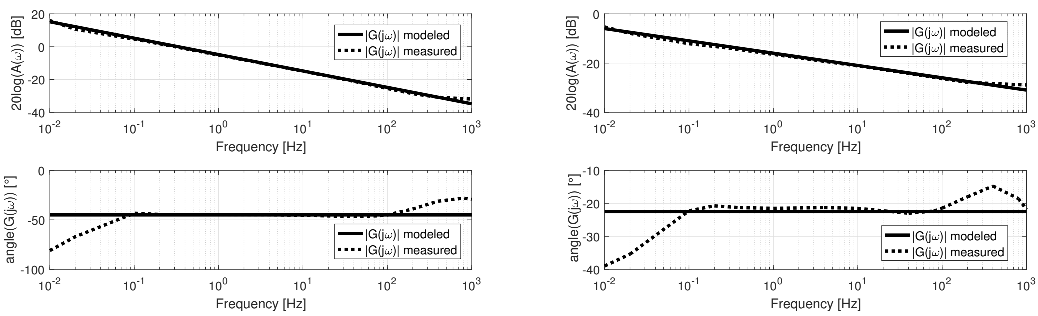


Fig. 1.8. Frequency responses of half- (*left*) and quarter (*right*)-order integrators.

to its numerical implementation presented in Fig. 1.9 gives an answer that such analog model posses a proper behavior until $t < 5$ s. (approximately), and then modeling error increases. The similar conclusion can be reached by analyzing the step response of quarter-order integrator and its numerical implementation shown in Fig. 1.10. The experimental curve overlaps numerical one until approximately 3 s, and after this time, the discrepancy increases.

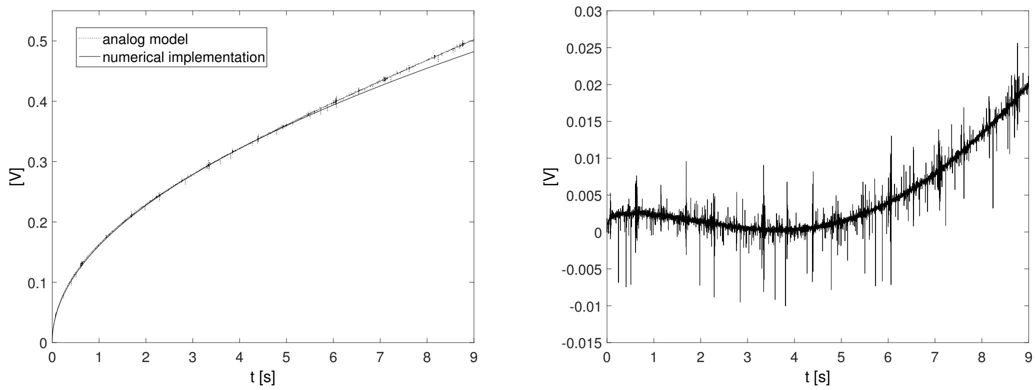


Fig. 1.9. Step responses of half-order integrator (*left*) and their modeling error (*right*). Input signal equals to $0.5 \cdot 1(t)$ V.

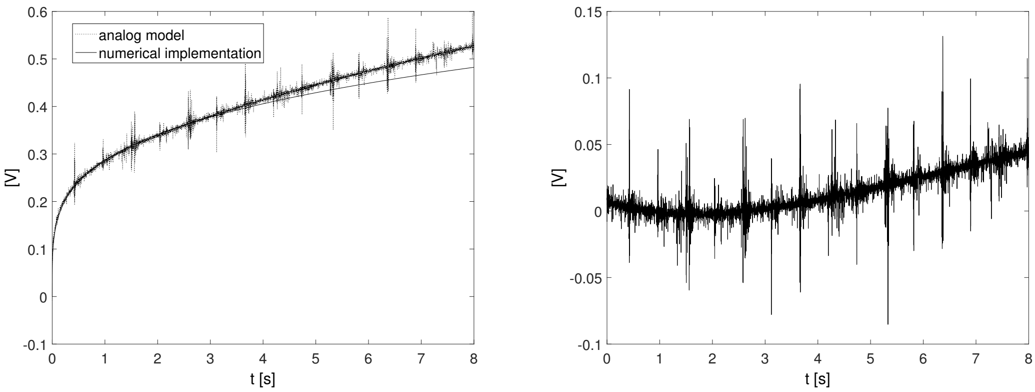


Fig. 1.10. Step response of quarter-order integrator (*left*) and their modeling error (*right*). Input signal equals to $0.05 \cdot 1(t)$ V.

Therefore, to avoid already mentioned constraints, appearing in analog modeling of fractional constant-order systems, all step responses of constant- and variable-order systems are time limited.

Chapter 2

Introduction to recursive fractional constant-order definition

In this chapter, the well known fractional constant-order Grünwald-Letnikov definition is remind, together with its matrix form. Moreover, the recursive fractional constant-order definition is introduced as well. As a novelty, the equivalence between fractional constant-order definition and recursive constant-order definition is proved based on their matrix forms. Then, the initial conditions in the form of finite and infinite time are mentioned and validated in experimental setup.

2.1 Fractional constant-order Grünwald-Letnikov definition

The fractional constant-order Grünwald-Letnikov definition is determined as a generalization of backward difference onto non-integer order. Thus, the well known Grünwald-Letnikov difference and derivative can be formulated:

Definition 2.1. The fractional constant-order Grünwald-Letnikov definition

$$ {}_0\mathrm{D}_t^{\alpha} f(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} f(t - jh), \tag{2.1} $$

where $\alpha \in \mathbb{R}$, $h > 0$ is a step time, $k = \lfloor t/h \rfloor$, and

$$ \binom{\alpha}{j} \equiv \begin{cases} 1 & \text{for} \quad j = 0\,, \\ \frac{\alpha(\alpha-1)\ldots(\alpha-j+1)}{j!} & \text{for} \quad j > 0\,. \end{cases} $$

Definition 2.2. The fractional constant-order Grünwald-Letnikov difference

$$ {}_0\Delta_l^{\alpha} f_l = \frac{1}{h^{\alpha}} \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} f_{l-j}, \tag{2.2} $$

for $l = 0, 1, 2, \ldots, k$.

Definition 2.1 corresponds to fractional constant-order derivative for $\alpha > 0$ or to fractional constant-order integral for $\alpha > 0$. In a special case, when $\alpha = 0$, it gives the original function $f(t)$.

Concerning the zero initial conditions and constant-orders α and β, the following orders composition property holds for Grünwald-Letnikov operator [55, 65]:

$$ {}_0\mathrm{D}_t^{\alpha}\left({}_0\mathrm{D}_t^{\beta} f(t)\right) = {}_0\mathrm{D}_t^{\beta}\left({}_0\mathrm{D}_t^{\alpha} f(t)\right) = {}_0\mathrm{D}_t^{\alpha+\beta} f(t). \tag{2.3}$$

2.1.1 The matrix form of fractional constant-order operator

The matrix form of the fractional order difference is given as follows [54, 56]:

$$\begin{pmatrix} {}_0\Delta_0^{\alpha} f_0 \\ {}_0\Delta_1^{\alpha} f_1 \\ {}_0\Delta_2^{\alpha} f_2 \\ \vdots \\ {}_0\Delta_k^{\alpha} f_k \end{pmatrix} = \begin{pmatrix} h^{-\alpha} & 0 & 0 & \ldots & 0 \\ v_{\alpha,1} & h^{-\alpha} & 0 & \ldots & 0 \\ v_{\alpha,2} & v_{\alpha,1} & h^{-\alpha} & \ldots & 0 \\ v_{\alpha,3} & v_{\alpha,2} & v_{\alpha,1} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ v_{\alpha,k} & v_{\alpha,k-1} & v_{\alpha,k-2} & \ldots & h^{-\alpha} \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_k \end{pmatrix},$$

where $v_{\alpha,i} = \frac{(-1)^i \binom{\alpha}{i}}{h^{\alpha}}$, and $h = t/k$, k is a number of samples.

Taking the limit $h \to \infty$ the following matrix form of fractional constant-order derivative is given:

$$\begin{pmatrix} {}_0\mathrm{D}_0^{\alpha} f(0) \\ {}_0\mathrm{D}_h^{\alpha} f(h) \\ {}_0\mathrm{D}_2^{\alpha} h f(2h) \\ \vdots \\ {}_0\mathrm{D}_k^{\alpha} h f(kh) \end{pmatrix} = \lim_{h\to\infty} W(\alpha, k) \begin{pmatrix} f(0) \\ f(h) \\ f(2h) \\ \vdots \\ f(kh) \end{pmatrix},$$

where

$$W(\alpha, k) = \begin{pmatrix} h^{-\alpha} & 0 & 0 & \ldots & 0 \\ v_{\alpha,h} & h^{-\alpha} & 0 & \ldots & 0 \\ v_{\alpha,2h} & v_{\alpha,h} & h^{-\alpha} & \ldots & 0 \\ v_{\alpha,3h} & v_{\alpha,2h} & v_{\alpha,h} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ v_{\alpha,kh} & v_{\alpha,(k-1)h} & v_{\alpha,(k-2)h} & \ldots & h^{-\alpha} \end{pmatrix}.$$

2.2 Recursive fractional constant-order operator

The fractional constant-order difference given by Def. 2.2 can be expressed by [75]:

$$\Delta_f^n(z) = \left(\frac{1 - z^{-1}}{h}\right)^{\alpha} F(z),$$

Then, it can be rewritten as

$$\Delta_f^n(z) = \frac{h^{-\alpha}}{(1 - z^{-1})^{\alpha}} F(z),$$

which gives the following relation

$$\Delta_f^n(z)(1 - z^{-1})^{-\alpha} = h^{-\alpha} F(z),$$

and can be represented in recursive form as

$$\sum_{j=0}^{k} (-1)^j \binom{-\alpha}{j} {}_0\Delta_k^{\alpha} f_{k-j} = h^{-\alpha} f_k.$$

Finally, it can be rewritten as a following definition:

Definition 2.3. The recursive fractional constant-order difference is as follows

$$ {}_0\Delta_k^{\alpha} f_k = h^{-\alpha} f_k - \sum_{j=1}^{k} (-1)^j \binom{-\alpha}{j} {}_0\Delta_k^{\alpha} f_{k-j}. \tag{2.4}$$

This type of difference is obtained for all values of previous differences.

For a continuous-time domain case, the recursive fractional order derivative can be described by [75]:

Definition 2.4. The recursive fractional constant-order derivative is as follows

$$ {}_0\mathrm{D}_t^{\alpha(t)} f(t) = \lim_{h \to 0} \left(\frac{f(t)}{h^{\alpha}} - \sum_{j=1}^{n} (-1)^j \binom{-\alpha}{j} {}_0\mathrm{D}_{t-jh}{}^{\alpha} f(t - jh) \right).$$

Having appropriate order and binomial coefficients dependency, the new recursive fractional variable-order derivatives can be formulated based on the recursive fractional constant-order definition.

2.3 Matrix approach for recursive fractional constant-order operator

By extension of Definition 2.3 there is possibility to achieve its matrix representation.

Lemma 2.5. *[88] For $\alpha = const$ the recursive fractional difference can be expressed in the following matrix form:*

$$\begin{pmatrix} {}_0\Delta_0^{\alpha} f_0 \\ {}_0\Delta_1^{\alpha} f_1 \\ {}_0\Delta_2^{\alpha} f_2 \\ \vdots \\ {}_0\Delta_k^{\alpha} f_k \end{pmatrix} = \mathfrak{Q}_0^k \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_k \end{pmatrix},$$

where

$$\mathfrak{Q}_0^k = \mathfrak{Q}(\alpha, k) \cdots \mathfrak{Q}(\alpha, 1) \mathfrak{Q}(\alpha, 0), \tag{2.5}$$

and

$$\mathfrak{Q}(\alpha, 0) = \left(\begin{array}{c|c} h^{-\alpha} & 0_{1,k} \\ \hline 0_{k,1} & I_{k,k} \end{array}\right) \in \mathbb{R}^{(k+1)\times(k+1)},$$

and for $r = 1, \dots, k$

$$\mathfrak{Q}(\alpha, r) = \left(\begin{array}{c|c|c} I_{r,r} & 0_{r,1} & 0_{r,k-r} \\ \hline q_r & h^{-\alpha} & 0_{1,k-r} \\ \hline 0_{k-r,r} & 0_{k-r,1} & I_{k-r,k-r} \end{array}\right) \in \mathbb{R}^{(k+1)\times(k+1)}, \tag{2.6}$$

where

$$q_r = (-w_{-\alpha,r}, -w_{-\alpha,r-1}, \dots, -w_{-\alpha,1}) \in \mathbb{R}^{1\times r},$$

and $w_{-\alpha,i} = (-1)^i \binom{-\alpha}{i}$, for $i = 1, \dots, r$, i.e., the jth entry of q_r is

$$(q_r)_j = -w_{-\alpha,r-j+1} = (-1)^{r-j+2} \binom{-\alpha}{r-j+1}, \quad \text{for} \quad j = 1, \dots, r.$$

Proof. It is obtained after consecutive evaluating of terms in Definition 2.3 for each time step $l = 0, 1, \dots, k$. First, for $l = 0$, we can write

$$\begin{pmatrix} {}_0\Delta_0^{\alpha} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_k \end{pmatrix} = \begin{pmatrix} h^{-\alpha} x_0 \\ f_1 \\ f_2 \\ \vdots \\ f_k \end{pmatrix} = \underbrace{\begin{pmatrix} h^{-\alpha} & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}}_{\mathfrak{Q}(\alpha,0)} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_k \end{pmatrix}.$$

Next, for $l = 1$:

$$\begin{pmatrix} {}_0\Delta_0^\alpha f_0 \\ {}_0\Delta_1^\alpha f_1 \\ f_2 \\ \vdots \\ f_k \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -w_{-\alpha,1} & h^{-\alpha} & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}}_{\mathfrak{Q}(\alpha,1)} \begin{pmatrix} {}_0\Delta_0^\alpha f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_k \end{pmatrix};$$

for $l = 2$:

$$\begin{pmatrix} {}_0\Delta_0^\alpha f_0 \\ {}_0\Delta_1^\alpha f_1 \\ {}_0\Delta_2^\alpha f_2 \\ f_3 \\ \vdots \\ f_k \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ -w_{-\alpha,2} & -w_{-\alpha,1} & h^{-\alpha} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}}_{\mathfrak{Q}(\alpha,2)} \begin{pmatrix} {}_0\Delta_0^\alpha f_0 \\ {}_0\Delta_1^\alpha f_1 \\ f_2 \\ \vdots \\ f_k \end{pmatrix};$$

and, generally, for $l = r$, we have

$$\begin{pmatrix} {}_0\Delta_0^\alpha f_0 \\ {}_0\Delta_1^\alpha f_1 \\ \vdots \\ {}_0\Delta_{r-1}^\alpha f_{r-1} \\ {}_0\Delta_r^\alpha f_r \\ f_{r+1} \\ \vdots \\ f_k \end{pmatrix} = \underbrace{\left(\begin{array}{c|c|c} I_{r,r} & 0_{r,1} & 0_{r,k-r} \\ \hline q_r & h^{-\alpha_r} & 0_{1,k-r} \\ \hline 0_{k-r,r} & 0_{k-r,1} & I_{k-r,k-r} \end{array}\right)}_{\mathfrak{Q}(\alpha,r)} \begin{pmatrix} {}_0\Delta_0^\alpha f_0 \\ {}_0\Delta_1^\alpha f_1 \\ \vdots \\ {}_0\Delta_{r-1}^\alpha f_{r-1} \\ f_r \\ \vdots \\ f_k \end{pmatrix}.$$

Finally, for $l = k$:

$$\begin{pmatrix} {}_0\Delta_0^\alpha f_0 \\ {}_0\Delta_1^\alpha f_1 \\ \vdots \\ {}_0\Delta_{k-1}^\alpha f_{k-1} \\ {}_0\Delta_k^\alpha f_k \end{pmatrix} = \underbrace{\left(\begin{array}{c|c} I_{k,k} & 0_{k,1} \\ \hline q_k & h^{-\alpha_k} \end{array}\right)}_{\mathfrak{Q}(\alpha,k)} \begin{pmatrix} {}_0\Delta_0^\alpha f_0 \\ {}_0\Delta_1^\alpha f_1 \\ \vdots \\ {}_0\Delta_{k-1}^\alpha f_{k-1} \\ f_k \end{pmatrix},$$

where $q_k = (-w_{-\alpha,k}, \dots, -w_{-\alpha,1})$.

Combining all this together, completes the proof. ■

Remark 2.6. Taking the limit $h \to 0$ to the above considerations, the following matrix form of recursive fractional constant-order derivative is given:

$$\begin{pmatrix} {}_0\mathrm{D}_0^{\alpha(t)} f(0) \\ {}_0\mathrm{D}_h^{\alpha(t)} f(h) \\ {}_0\mathrm{D}_{2h}^{\alpha(t)} f(2h) \\ \vdots \\ {}_0\mathrm{D}_{kh}^{\alpha(t)} f(kh) \end{pmatrix} = \lim_{h\to 0} \mathfrak{Q}_0^k \begin{pmatrix} f(0) \\ f(h) \\ f(2h) \\ \vdots \\ f(kh) \end{pmatrix}. \tag{2.7}$$

Although, the recursive fractional constant-order derivative given by Definition 2.4 origin, from well-known Grïwald-Letnikov definition given by Definition 2.1 what implies their equivalence, it is an interesting issue to prove this property based on their matrix forms. In this purpose let us formulate Lemma 2.7 and Lemma 2.8.

Lemma 2.7. *The product of matrix form equivalent to the fractional-constant-order Grünwald-Letnikov derivative and its recursive matrix form for negative value of orders gives an identity matrix*

$$W(-\alpha, k) \prod_{j=0}^{k} \mathfrak{Q}(\alpha, k-j) = I_{k+1,k+1}. \tag{2.8}$$

Proof. Let us write the matrix form of Grünwald-Letnikov derivative for negative value of orders, then

$$W(-\alpha,k) = \begin{pmatrix} h^\alpha & 0 & 0 & \dots & 0 & 0 \\ h^\alpha w_{-\alpha,1} & h^\alpha & 0 & \dots & 0 & 0 \\ h^\alpha w_{-\alpha,2} & h^\alpha w_{-\alpha,1} & h^\alpha & \dots & 0 & 0 \\ h^\alpha w_{-\alpha,3} & h^\alpha w_{-\alpha,2} & h^\alpha w_{-\alpha,1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ h^\alpha w_{-\alpha,k-1} & h^\alpha w_{-\alpha,k-2} & h^\alpha w_{-\alpha,k-3} & \dots & h^\alpha & 0 \\ h^\alpha w_{-\alpha,k} & h^\alpha w_{-\alpha,k-1} & h^\alpha w_{-\alpha,k-2} & \dots & h^\alpha w_{-\alpha,1} & h^\alpha \end{pmatrix}.$$

The first matrix product can be described as the following block matrices:

$$\begin{aligned} W(-\alpha,k)\mathfrak{Q}(\alpha,k) &= \left(\begin{array}{c|c} W(-\alpha,k-1) & 0_{k,1} \\ \hline R(-\alpha,k) & h^\alpha \end{array}\right) \left(\begin{array}{c|c} I_{k,k} & 0_{k,1} \\ \hline q_{1,k} & h^{-\alpha} \end{array}\right) \\ &= \left(\begin{array}{c|c} W(-\alpha,k-1) & 0_{k,1} \\ \hline 0_{1,k} & 1 \end{array}\right), \end{aligned}$$

where

$$
\begin{aligned}
R(-\alpha, i) &= \begin{pmatrix} h^{\alpha} w_{-\alpha,i} & h^{\alpha} w_{-\alpha,i-1} & \cdots & h^{\alpha} w_{-\alpha,1} \end{pmatrix}, \\
h^{\alpha} q_{1,i} &= \left(-h^{\alpha} w_{-\alpha,i}, -h^{\alpha} w_{-\alpha,i-1}, \ldots, -h^{\alpha} w_{-\alpha,1}\right).
\end{aligned}
$$

The second matrix product can be expressed by

$$
\begin{aligned}
&W(-\alpha, k)\mathfrak{Q}(\alpha, k)\mathfrak{Q}(\alpha, k-1) \\
&= \left(\begin{array}{cc|c} W(-\alpha, k-2) & 0_{k-1,1} & 0_{k-1,1} \\ R(-\alpha, k-1) & h^{\alpha} & 0 \\ \hline 0_{1,k-1} & 0 & 1 \end{array}\right) \left(\begin{array}{cc|c} I_{k-1,k-1} & 0_{k-1,1} & 0_{k-1,1} \\ q_{1,k-1} & h^{-\alpha} & 0 \\ \hline 0_{1,k-1} & 0 & 1 \end{array}\right) \\
&= \left(\begin{array}{cc|c} W(-\alpha, k-2) & 0_{k-1,1} & 0_{k-1,1} \\ 0_{1,k-1} & 1 & 0 \\ \hline 0_{1,k-1} & 0 & 1 \end{array}\right).
\end{aligned}
$$

For general case, when $j = r$ we have the following matrix product

$$
\begin{aligned}
&\left(W(-\alpha, k)\prod_{j=0}^{r-1} \mathfrak{Q}(\alpha, k-j)\right)\mathfrak{Q}(\alpha, k-r) \\
&= \left(\begin{array}{cc|c} W(-\alpha, k-r-1) & 0_{k-r,1} & 0_{k-r,r} \\ R(-\alpha, k-r) & h^{\alpha} & 0 \\ \hline 0_{r,k-r} & 0_{r,1} & I_{r,r} \end{array}\right) \left(\begin{array}{cc|c} I_{k-r,k-r} & 0_{k-r,1} & 0_{k-r,r} \\ q_{1,k-r} & h^{-\alpha} & 0_{1,r} \\ \hline 0_{r,k-r} & 0_{r,1} & I_{r,r} \end{array}\right) \\
&= \left(\begin{array}{cc|c} W(-\alpha, k-r-1) & 0_{k-r,1} & 0_{k-r,r} \\ 0_{1,k-r} & 1 & 0 \\ \hline 0_{r,k-r} & 0_{r,1} & I_{r,r} \end{array}\right).
\end{aligned}
$$

Finally, for $j = k$ matrix product equals to

$$
\begin{aligned}
\left(W(-\alpha, k)\prod_{j=0}^{k-1} \mathfrak{Q}(\alpha, k-j)\right)\mathfrak{Q}(\alpha, 0) &= \left(\begin{array}{c|c} h^{\alpha} & 0_{1,k} \\ \hline 0_{k,1} & I_{k,k} \end{array}\right) \left(\begin{array}{c|c} h^{-\alpha} & 0_{1,k} \\ \hline 0_{k,1} & I_{k,k} \end{array}\right) \\
&= \left(\begin{array}{c|c} 1 & 0_{1,k} \\ \hline 0_{k,1} & I_{k,k} \end{array}\right).
\end{aligned}
$$

Consecutive multiplying all of terms from Lemma 2.7 ends the proof. ■

Lemma 2.8. *The product of matrix forms equivalent to the fractional constant-order derivative with negative value of orders gives an identity matrix.*

$$
W(-\alpha, k)W(\alpha, k) = I_{k+1,k+1}.
$$

Proof. The proof implied from the order composition property for fractional constant-order derivative with zero initial condition. ■

Finally, the following theorem can be written:

Theorem 2.9. *The fractional constant-order definition is equivalent to its recursive form.*

Proof. The proof is a conclusion directly based on Lemma 2.7 and Lemma 2.8.

$$W(-\alpha,k)\prod_{j=0}^{k}\mathfrak{Q}(\alpha,k-j)=W(-\alpha,k)W(\alpha,k) \implies \prod_{j=0}^{k}\mathfrak{Q}(\alpha,k-j)=W(\alpha,k).$$

■

2.4 Initial conditions for recursive fractional constant-order operator

The initial value of voltage occurring at the beginning of half- or quarter-order domino-ladders plays a significant role during experimental approach. The initial voltage is not distributed evenly to the rest of branches and obviously, has an impact on experimental results. This is a main motivation for providing the method for including initial conditions into recursive fractional order definition. In this method the voltages of all capacitors are equal to some initial value of voltage. It can be caused by charging the impedance structure until all capacitors achieved the desire value of voltage. To provide the method for including initial conditions in above sense let us assume the fractional order recursive difference for time k^* [71, 91]:

$$ {}_0\Delta^{\alpha}_{k*}x_{k^*}=x_{k^*}-\sum_{j=1}^{k^*}(-1)^j\binom{-\alpha}{j}{}_0\Delta^{\alpha}_{k^*-j}x_{k^*-j}. \tag{2.9}$$

The beginning time in (2.9) is shifted to the point $t^*=T$. This means, that the system where the algorithm is applied, has non-zero energy. In this case, the difference can be rewritten for $k=k^*-T$ in the following form [71, 91]:

$$ {}_{-T}\Delta^{\alpha}_{k}x_{k}=x_{k}-\sum_{j=1}^{k+T}(-1)^j\binom{-\alpha}{j}{}_{-T}\Delta^{\alpha}_{k-j}x_{k-j}\,. \tag{2.10}$$

The sum occurring in (2.10) can be divided onto two parts

$$ {}_{-T}\Delta^{\alpha}_{k}x_{k}=x_{k}-\sum_{j=1}^{k-1}(-1)^j\binom{-\alpha}{j}{}_{-T}\Delta^{\alpha}_{k-j}x_{k-j}-\sum_{j=k}^{k+T}(-1)^j\binom{-\alpha}{j}{}_{-T}\Delta^{\alpha}_{k-j}x_{k-j}\,. \tag{2.11}$$

Thus, we have the iterated differences (for new time k) from $\Delta^\alpha x_{k-1}$ to $\Delta^\alpha x_1$ given by first sum and differences from $\Delta^\alpha x_0$ to $\Delta^\alpha x_{-T}$ given by second sum. As it can be notice, for a practical case, proper initial conditions statement (in opposition to integer order system) is a series from $\Delta^\alpha x_0$ to $\Delta^\alpha x_{-T}$.

Constant, finite-time conditions case

When the system is in a steady state, before for example control algorithm starts ($t < 0$), what is easy to imagine in practical applications, we can assume that ${}_T\Delta^\alpha_i x_i = c = const$ for $i = -T \dots 0$. In this case, (2.11) has the following form [71, 91]:

$$ {}_{-T}\Delta^\alpha_k x_k \quad = \quad x_k - \sum_{j=1}^{k-1} (-1)^j \binom{-\alpha}{j} {}_{-T}\Delta^\alpha_k x_{k-j} - \sum_{j=k}^{k+T} (-1)^j \binom{-\alpha}{j} c\,. $$

Remark 2.10. *We have the following relation*

$$ \sum_{j=0}^{\infty} (-1)^j \binom{-\alpha}{j} = 0\,, $$

which can be proved by taking into consideration that

$$ \sum_{j=0}^{\infty} (-1)^j \binom{-\alpha}{j} z^k = (1-z)^\alpha\,, $$

and substitute $z = 1$ to the relation above.

Let us now analyze the ideal situation for infinite length of initial conditions function $T \to \infty$, and autonomous system with $x_k = 0$. For such a situation, the first time step of difference is given as follows [71, 91]

$$ {}_{-\infty}\Delta^\alpha_1 x_1 = -\sum_{j=1}^{\infty} (-1)^j \binom{-\alpha}{j} c\,, $$

which can be rewritten in the form of

$$ {}_{-\infty}\Delta^\alpha_1 x_1 = c - \sum_{j=0}^{\infty} (-1)^j \binom{-\alpha}{j} c\,. $$

Based on Remark 2.10, it can be evaluated into the following form

$$ {}_{-\infty}\Delta^\alpha_1 x_1 = c,\,. $$

We obtain expected result that for infinite length of initial conditions, system without additional input (in the form of x_k) will stay in the same point.

For example, let us assume the ideal domino ladder that is analog realization of half-order integrator. When all its capacitors are charged to the same voltage, the initial conditions are equivalent to infinite length constant initial conditions. Without additional voltage source, the domino ladder should keep the voltage on capacitors.

Let us study the case of finite length of initial conditions. In such a case the first step of difference has the following form [71, 91]

$$_{-T}\Delta_1^{\alpha} x_1 = -\sum_{j=1}^{T+1} (-1)^j \binom{-\alpha}{j} c \,,$$

which can be rewritten, similarly to infinite length case, as follows

$$_{-T}\Delta_1^{\alpha} x_1 = c + \sum_{j=T+2}^{\infty} (-1)^j \binom{-\alpha}{j} c.$$

As it can be seen (also in Fig. 2.1), this time the value of difference is not equal to value of initial constant conditions.

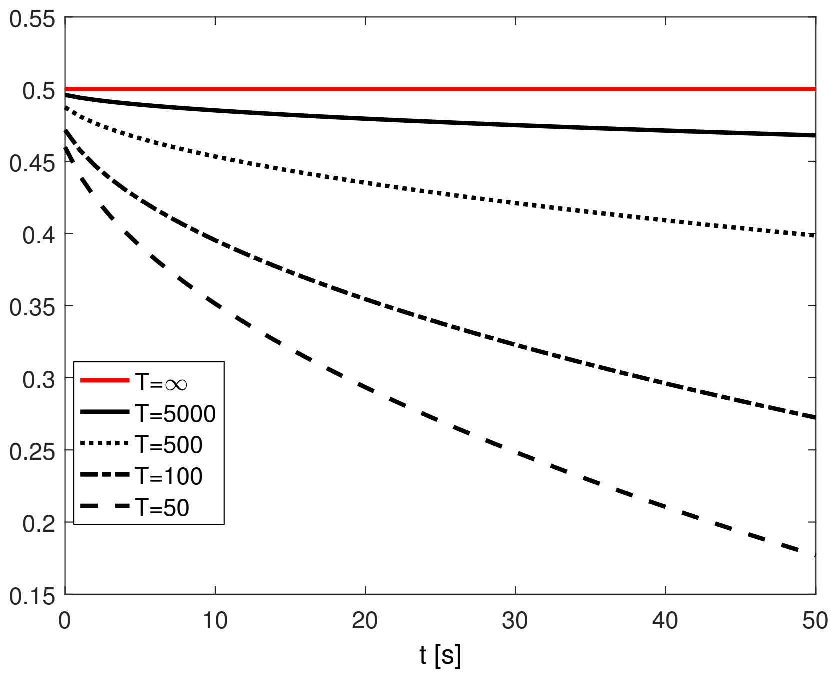


Fig. 2.1. Results of $\alpha = 0.5$ integrators output with initial conditions $c = 0.5$ and different values of T

Constant, infinite time initial conditions case

For infinite length of the series of initial conditions this can be rewritten in the form of the following theorem [71, 91]:

Theorem 2.11. *Fractional order difference for initial conditions in the form of infinite length constant function $\Delta^\alpha x_i = c = const$ for $i = -\infty \ldots 0$ is given by the following relation:*

$$ {}_{-\infty}\Delta^\alpha_l x_l = \frac{x_l}{h^\alpha} - \sum_{j=1}^{l-1} (-1)^j \binom{-\alpha}{j} \left({}_{-\infty}\Delta^\alpha_{l-j} x_{l-j} - c \right) + c \tag{2.12} $$

for $l = 1, 2, \ldots, k$.

Proof. The second sum in the (2.11), for infinite length of initial conditions $T \to \infty$ (which implies that also $T + k \to \infty$), can be obtained in the following form

$$ \sum_{j=0}^{\infty} (-1)^j \binom{-\alpha}{j} c = \sum_{j=0}^{k-1} (-1)^j \binom{-\alpha}{j} c + \sum_{j=k}^{k+T} (-1)^j \binom{-\alpha}{j} c \,. $$

Using relation given by Remark 2.10 we obtain

$$ \begin{aligned} \sum_{j=k}^{k+T} (-1)^j \binom{-\alpha}{j} c &= \sum_{j=0}^{\infty} (-1)^j \binom{-\alpha}{j} c - \sum_{j=0}^{k-1} (-1)^j \binom{-\alpha}{j} c \\ &= -c - \sum_{j=1}^{k-1} (-1)^j \binom{-\alpha}{j} c \,, \end{aligned} $$

which combining with Remark 2.10 ends the proof. ∎

Continuous-time case

For a continuous-time domain case, the fractional order difference definition can be rewritten in the following form [71, 91]:

Theorem 2.12. *Fractional order derivative for initial conditions in the form of infinite length constant function ${}_{-\infty}\mathrm{D}^\alpha_l f(t) = c = const.$ for $l = (-\infty, 0)$ is given by the following relation:*

$$ {}_{-\infty}\mathrm{D}^\alpha_t f(t) = \lim_{h \to 0} \left[\frac{f(t)}{h^\alpha} - \sum_{j=1}^{n} (-1)^j \binom{-\alpha}{j} ({}_{-\infty}\mathrm{D}^\alpha_{t-jh} f(t) - c) + c \right] . $$

Proof. The proof is fully analogical to the proof of Theorem 2.11. ∎

2.5 Experimental results

Realization of the fractional constant-order inertial system based on fractional constant-order integrator presented in Fig. 1.7 is shown in Fig. 2.2. To gather data for fractional constant-order inertial system with initial conditions, the domino-ladder structure is firstly loaded to the initial value of voltage and then the system presented in Fig. 2.2 is running with $u(t) = 0$. The initial value of voltage for all measurements equals to $0.2V$ and the sample time equals to 0.001 s. The convenience Matlab/Simulink variable-order toolbox [80] is used to obtain the numerical implementations of variable-order systems during all experiments presented in this thesis.

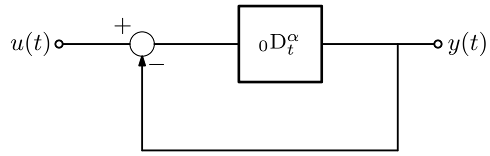


Fig. 2.2. Realization of the fractional variable-order inertial system based on fractional constant-order integrator (see Fig. 1.7).

Example 2.13. [71] The $\alpha = -0.5$ order inertial system with initial condition.

In this case the Z impedance presented in Fig. 1.7 takes the structure of half-order impedance presented in Section 1.6.1 and resistor $R = 5\text{k}\Omega$. Identified system has the following form:

$$y(t) = 20{}_{-\infty}\mathrm{D}_t^{-0.5}(u(t) - y(t)).$$

The experimental results of fractional constant-order inertial system with order -0.5, compared to their numerical results are presented in Fig. 2.3.

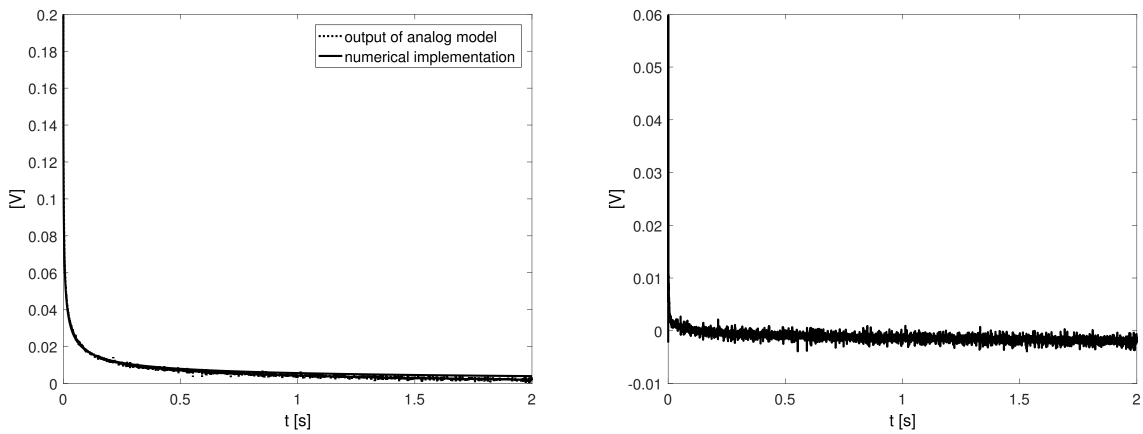


Fig. 2.3. Results of analog and numerical implementation of fractional constant-order inertial system with order -0.5 and initial condition equals to 0.2V (*left*), their modeling error (*right*).

Example 2.14. [71] The $\alpha = -0.25$ order inertial system with initial condition.

In this case the Z impedance presented in Fig. 1.7 takes the structure of quarter-order impedance presented in Section 1.6.1 and resistor $R = 5\text{k}\Omega$.

Identified system has the following form:

$$y(t) = 1.55{}_{-\infty}\mathrm{D}_t^{-0.25}(u(t) - y(t)),$$

The experimental results of fractional order inertial system with order -0.25, compared to the numerical results, are presented in Fig. 2.4.

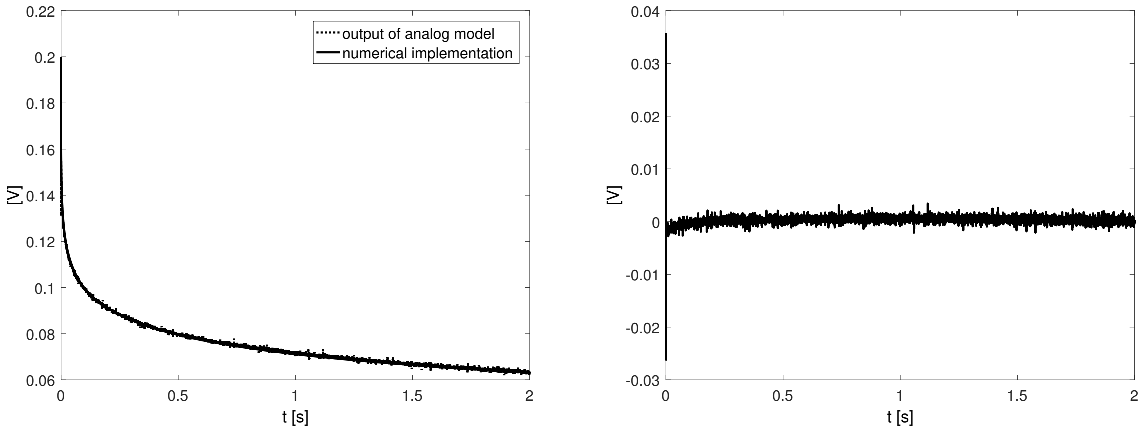


Fig. 2.4. Results of analog and numerical implementation of fractional order inertial system with order -0.25 and initial condition equals to 0.2V (*left*), their modeling error (*right*).

2.6 Summary

In this chapter, the Grünwald-Letnikov fractional constant-order definition and its recursive form were remind. As a novelty, the equivalence between Grünwald-Letnikov definition and its recursive form was proved based on their matrix forms. Next, the initial condition case for recursive definition was considered in the form of finite and infinite time. At the end, the initial conditions for fractional constant-order integral systems were experimentally validated and show high accuracy of proposed method.

Chapter 3

The $\mathcal{A}$-type fractional variable-order definition

In this chapter, the mostly used in many implementations, $\mathcal{A}$-type fractional variable-order difference and derivative are presented together with their matrix forms. Moreover, the interpretation of such definition based on its equivalent switching order scheme is introduced and experimentally validated.

The $\mathcal{A}$-type operator can be successfully used in control systems. The practical application where such definition was applied to control the electromagnetic servo was published in [50].

3.1 Introduction to $\mathcal{A}$-type fractional variable-order definition

The $\mathcal{A}$-type fractional variable-order operator owes its origin to well known Grünwald-Letnikov definition. Consider the case when constant-order is extended to time-varying. In this approach, all samples are calculated with present value of order. The $\mathcal{A}$-type operator has the following form:

Definition 3.1. [26, 104] The $\mathcal{A}$-type fractional variable-order derivative is as follows

$$ {}_{0}^{\mathcal{A}}\mathrm{D}_{t}^{\alpha(t)} f(t) = \lim_{h\to 0} \sum_{j=0}^{n} \frac{(-1)^j}{h^{\alpha(t)}} \binom{\alpha(t)}{j} f(t - jh), $$

where $n = \lfloor t/h \rfloor$ and h is a sampling time.

Definition 3.2. [26, 104] The discrete realization of the $\mathcal{A}$-type fractional variable-order derivative is as follows

$$ {}_{0}^{\mathcal{A}}\Delta_{l}^{\alpha_l} f_l = \sum_{j=0}^{k} \frac{(-1)^j}{h^{\alpha_l}} \binom{\alpha_l}{j} f_{l-j}, $$

for $l = 0, 1, 2, \ldots, k$.

In Fig. 3.1, plots of step function derivatives (according to Def. 3.1) are presented for $\alpha_1 = -1$, $\alpha_2 = -2$, and

$$\alpha_3(t) = \begin{cases} -1 & \text{for } 0 \leq t < 1, \\ -2 & \text{for } 1 \leq t \leq 2. \end{cases} \tag{3.1}$$

Considering the step response of the $\mathcal{A}$-type derivative given in Fig. 3.1, it is noticeable

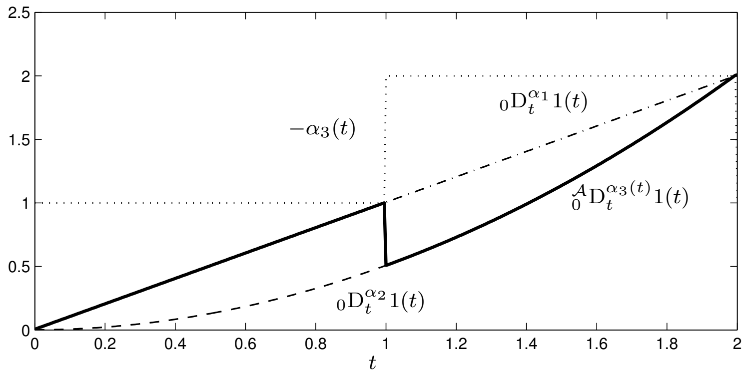


Fig. 3.1. Plots of step function derivatives with respect to the $\mathcal{A}$-type derivative (given by Def. 3.1).

that until switching time the output of $\mathcal{A}$-type derivative refers to derivative of order -1. After switching time, the output of $\mathcal{A}$-type operator rapidly goes to the output of system being representing by derivative of order -2. Therefore, such variable-order derivative can be interpreted as switching order systems.

3.1.1 Matrix form of the $\mathcal{A}$-type fractional variable-order derivative

The matrix form of the $\mathcal{A}$-type fractional variable-order difference is given by [73]

$$\begin{pmatrix} {}^{\mathcal{A}}_0\Delta_0^{\alpha_0} f_0 \\ {}^{\mathcal{A}}_0\Delta_1^{\alpha_1} f_1 \\ {}^{\mathcal{A}}_0\Delta_2^{\alpha_2} f_2 \\ {}^{\mathcal{A}}_0\Delta_3^{\alpha_3} f_3 \\ \vdots \\ {}^{\mathcal{A}}_0\Delta_k^{\alpha_k} f_k \end{pmatrix} = \begin{pmatrix} h^{-\alpha_0} & 0 & 0 & \dots & 0 \\ w_{\alpha_1,1} & h^{-\alpha_1} & 0 & \dots & 0 \\ w_{\alpha_2,2} & w_{\alpha_2,1} & h^{-\alpha_2} & \dots & 0 \\ w_{\alpha_3,3} & w_{\alpha_3,2} & w_{\alpha_3,1} & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ w_{\alpha_k,k} & w_{\alpha_k,k-1} & w_{\alpha_3,k-2} & \dots & h^{-\alpha_k} \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_k \end{pmatrix},$$

Taking the limit $h \to 0$, the following matrix form of the $\mathcal{A}$-type derivative is given [86]:

$$\begin{pmatrix} {}^{\mathcal{A}}_0\mathrm{D}_0^{\alpha(t)} f(0) \\ {}^{\mathcal{A}}_0\mathrm{D}_h^{\alpha(t)} f(h) \\ {}^{\mathcal{A}}_0\mathrm{D}_{2h}^{\alpha(t)} f(2h) \\ \vdots \\ {}^{\mathcal{A}}_0\mathrm{D}_{kh}^{\alpha(t)} f(kh) \end{pmatrix} = \lim_{h \to 0} {}^{\mathcal{A}}W(\alpha, k) \begin{pmatrix} f(0) \\ f(h) \\ f(2h) \\ \vdots \\ f(kh) \end{pmatrix}, \tag{3.2}$$

where

$$
{}^{\mathcal{A}}W(\alpha,k) = \begin{pmatrix} h^{-\alpha(0)} & 0 & 0 & \ldots & 0 \\ w_{\alpha(h),1} & h^{-\alpha(h)} & 0 & \ldots & 0 \\ w_{\alpha(2h),2} & w_{\alpha(2h),1} & h^{-\alpha(2h)} & \ldots & 0 \\ w_{\alpha(3h),3} & w_{\alpha(3h),2} & w_{\alpha(3h),1} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ w_{\alpha(kh),k} & w_{\alpha(kh),k-1} & w_{\alpha(kh),k-2} & \ldots & h^{-\alpha(kh)} \end{pmatrix}. \tag{3.3}
$$

3.1.2 Numerical scheme for the $\mathcal{A}$-type operator

Let us introduce the following switching scheme presented in Fig. 3.2 based on the chain of derivatives blocks. The switches S_i, $i = 0, \ldots, k$, take the following positions

$$
S_i = \begin{cases} a & \text{for } t \geq (i+1)h, \\ b & \text{for } t \in [0,(i+1)h), \end{cases} \qquad i = 0, \ldots, k.
$$

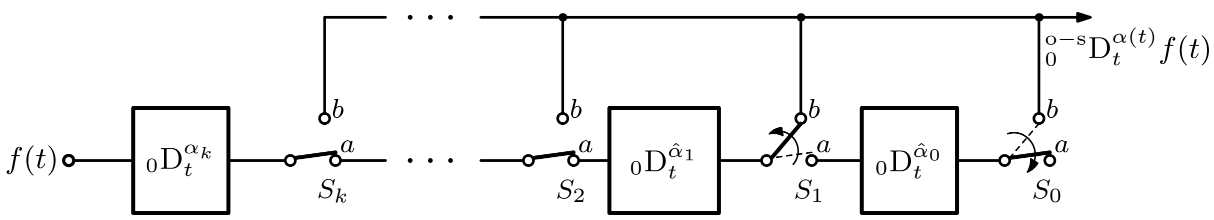


Fig. 3.2. Structure of the multiple output-switching order derivative ${}_0^{\text{o-s}}\mathrm{D}_t^{\alpha(t)}f(t)$ (configuration after switch between orders α_0 and α_1, i.e., in time $t \in (0,h)$)

Based on Fig. 3.2 we have the following result.

Lemma 3.3. *[88] The numerical description of the multiple output-switching scheme, when we switch orders $\alpha_0, \ldots, \alpha_k$ in every ih time instant, is the following:*

$$
\begin{pmatrix} {}_0^{\text{o-s}}\mathrm{D}_0^{\alpha(t)}f(0) \\ {}_0^{\text{o-s}}\mathrm{D}_h^{\alpha(t)}f(h) \\ \vdots \\ {}_0^{\text{o-s}}\mathrm{D}_{kh}^{\alpha(t)}f(kh) \end{pmatrix} = \lim_{h\to 0} \left(\prod_{i=0}^{k} W(\hat{\alpha}_i,k,i) \right) \begin{pmatrix} f(0) \\ f(h) \\ \vdots \\ f(kh) \end{pmatrix}, \tag{3.4}
$$

where

$$
W(\hat{\alpha}_i,k,i) = \begin{pmatrix} W(\hat{\alpha}_i,i) & 0_{i+1,k-i} \\ 0_{k-i,i+1} & I_{k-i,k-i} \end{pmatrix}, \quad \hat{\alpha}_i = \alpha_i - \alpha_{i+1}, \quad i = 0, \ldots, k-1,
$$

and

$$\alpha(t) = \begin{cases} \alpha_{i+1} + \hat{\alpha}_i & \textit{for} \quad 0 \le t < (i+1)h, \\ \alpha_{i+1} & \textit{for} \quad t \ge (i+1)h, \end{cases} \qquad i = 0, \ldots, k-1.$$

Proof. The output signal after the α_k block has the following form:

$$\begin{pmatrix} {}_0^{\text{o-s}}\mathrm{D}_0^{\alpha_k} f(0) \\ {}_0^{\text{o-s}}\mathrm{D}_h^{\alpha_k} f(h) \\ {}_0^{\text{o-s}}\mathrm{D}_{2h}^{\alpha_k} f(2h) \\ \vdots \\ {}_0^{\text{o-s}}\mathrm{D}_{(k-1)h}^{\alpha_k} f((k-1)h) \\ {}_0^{\text{o-s}}\mathrm{D}_{kh}^{\alpha_k} f(kh) \end{pmatrix} = \lim_{h\to 0} W(\alpha_k, k) \begin{pmatrix} f(0) \\ f(h) \\ f(2h) \\ \vdots \\ f((k-1)h) \\ f(kh) \end{pmatrix}.$$

The additional block $\hat{\alpha}_{k-1}$ is connected in whole time despite of time k, that is why this signal is represented by the matrix $W(\hat{\alpha}_{k-1}, k, k-1)$. At the output of this block we obtain derivative of order α_{k-1} until time k when the order is equal to α_k

$$\begin{pmatrix} {}_0^{\text{o-s}}\mathrm{D}_0^{\alpha_{k-1}} f(0) \\ {}_0^{\text{o-s}}\mathrm{D}_h^{\alpha_{k-1}} f(h) \\ {}_0^{\text{o-s}}\mathrm{D}_{2h}^{\alpha_{k-1}} f(2h) \\ \vdots \\ {}_0^{\text{o-s}}\mathrm{D}_{(k-1)h}^{\alpha_{k-1}} f((k-1)h) \\ {}_0^{\text{o-s}}\mathrm{D}_{kh}^{\alpha_k} f(kh) \end{pmatrix} = \lim_{h\to 0} W(\hat{\alpha}_{k-1}, k, k-1) \begin{pmatrix} {}_0^{\text{o-s}}\mathrm{D}_0^{\alpha_k} f(0) \\ {}_0^{\text{o-s}}\mathrm{D}_h^{\alpha_k} f(h) \\ {}_0^{\text{o-s}}\mathrm{D}_{2h}^{\alpha_k} f(2h) \\ \vdots \\ {}_0^{\text{o-s}}\mathrm{D}_{(k-1)h}^{\alpha_k} f((k-1)h) \\ {}_0^{\text{o-s}}\mathrm{D}_{kh}^{\alpha_k} f(kh) \end{pmatrix}.$$

Repeating analogously, we get the output signal from the block of derivative $\hat{\alpha}_1$ in the following form

$$\begin{pmatrix} {}_0^{\text{o-s}}\mathrm{D}_0^{\alpha_1} f(0) \\ {}_0^{\text{o-s}}\mathrm{D}_h^{\alpha_1} f(h) \\ {}_0^{\text{o-s}}\mathrm{D}_{2h}^{\alpha_2} f(2h) \\ \vdots \\ {}_0^{\text{o-s}}\mathrm{D}_{(k-1)h}^{\alpha_{k-1}} f((k-1)h) \\ {}_0^{\text{o-s}}\mathrm{D}_{kh}^{\alpha_k} f(kh) \end{pmatrix} = \lim_{h\to 0} W(\hat{\alpha}_1, k, 1) \begin{pmatrix} {}_0^{\text{o-s}}\mathrm{D}_0^{\alpha_2} f(0) \\ {}_0^{\text{o-s}}\mathrm{D}_h^{\alpha_2} f(h) \\ {}_0^{\text{o-s}}\mathrm{D}_{2h}^{\alpha_2} f(2h) \\ \vdots \\ {}_0^{\text{o-s}}\mathrm{D}_{(k-1)h}^{\alpha_{k-1}} f((k-1)h) \\ {}_0^{\text{o-s}}\mathrm{D}_{kh}^{\alpha_k} f(kh) \end{pmatrix}.$$

Finally, we obtain the output signal from the block of derivative $\hat{\alpha}_0$ in the following form:

$$\begin{pmatrix} {}_0^{\text{o-s}}\mathrm{D}_0^{\alpha_0} f(0) \\ {}_0^{\text{o-s}}\mathrm{D}_h^{\alpha_1} f(h) \\ \vdots \\ {}_0^{\text{o-s}}\mathrm{D}_{(k-1)h}^{\alpha_{k-1}} f((k-1)h) \\ {}_0^{\text{o-s}}\mathrm{D}_{kh}^{\alpha_k} f(kh) \end{pmatrix} = \lim_{h\to 0} W(\hat{\alpha}_0, k, 0) \begin{pmatrix} {}_0^{\text{o-s}}\mathrm{D}_0^{\alpha_1} f(0) \\ {}_0^{\text{o-s}}\mathrm{D}_h^{\alpha_1} f(h) \\ \vdots \\ {}_0^{\text{o-s}}\mathrm{D}_{(k-1)h}^{\alpha_{k-1}} f((k-1)h) \\ {}_0^{\text{o-s}}\mathrm{D}_{kh}^{\alpha_k} f(kh) \end{pmatrix}.$$

Combining all this together, we get (3.4), completing the proof. ∎

Theorem 3.4. *[88] Matrix approach of the $\mathcal{A}$-type derivative given by* (3.2) *is equivalent to the output-switching scheme given by* (3.4)*, i.e.,*

$${}_0^{\mathcal{A}}\mathrm{D}_t^{\alpha(t)} f(t) \equiv {}_0^{\text{o-s}}\mathrm{D}_t^{\alpha(t)} f(t).$$

Proof. For simplicity, let us assume the case of one switch between orders, say α_1 and α_2 occurring at time $T = \tau h$, $\tau \in \mathbb{N}_+$, we have the following matrix form based on Lemma 3.3:

$$\begin{pmatrix} {}_0^{\text{o-s}}\mathrm{D}_0^{\alpha(t)} f(0) \\ \vdots \\ {}_0^{\text{o-s}}\mathrm{D}_T^{\alpha(t)} f(T) \\ \vdots \\ {}_0^{\text{o-s}}\mathrm{D}_{kh}^{\alpha(t)} f(kh) \end{pmatrix} = \lim_{h\to 0} W(\hat{\alpha}_1, k, \tau-1) W(\alpha_2, k) \begin{pmatrix} f(0) \\ \vdots \\ f(T) \\ \vdots \\ f(kh) \end{pmatrix}, \tag{3.5}$$

where

$$W(\hat{\alpha}_1, k, \tau-1) = \begin{pmatrix} W(\hat{\alpha}_1, \tau-1) & 0_{\tau,k-\tau+1} \\ 0_{k-\tau+1,\tau} & I_{k-\tau+1,k-\tau+1} \end{pmatrix}, \qquad \hat{\alpha}_1 = \alpha_1 - \alpha_2,$$

and

$$\alpha(t) = \begin{cases} \alpha_2 + \hat{\alpha}_1 & \text{for } t < T, \\ \alpha_2 & \text{for } t \geq T. \end{cases}$$

The matrix product

$$\begin{aligned} W(\hat{\alpha}_1, k, \tau-1) W(\alpha_2, k) &= \begin{pmatrix} W(\hat{\alpha}_1, \tau-1) & 0_{\tau,k-\tau+1} \\ 0_{k-\tau+1,\tau} & I_{k-\tau+1,k-\tau+1} \end{pmatrix} \begin{pmatrix} W(\alpha_2, \tau-1) & 0_{\tau,k-\tau+1} \\ A(\alpha_2) & B(\alpha_2) \end{pmatrix} \\ &= \begin{pmatrix} W(\hat{\alpha}_1 + \alpha_2, \tau-1) & 0_{\tau,k-\tau+1} \\ A(\alpha_2) & B(\alpha_2) \end{pmatrix} \\ &= \begin{pmatrix} W(\alpha_1, \tau-1) & 0_{\tau,k-\tau+1} \\ A(\alpha_2) & B(\alpha_2) \end{pmatrix}, \end{aligned}$$

where $A(\alpha_2) \in \mathbb{R}^{(k-\tau+1)\times\tau}$ and $B(\alpha_2) \in \mathbb{R}^{(k-\tau+1)\times(k-\tau+1)}$ are suitable sub-matrices of $W(\alpha_2, k)$, obviously corresponds to ${}^{\mathcal{A}}W(\alpha, k)$ given by (3.3) for

$$\alpha = \alpha(t) = \begin{cases} \alpha_1 & \text{for } t < T, \\ \alpha_2 & \text{for } t \geq T, \end{cases}$$

i.e., in (3.3) we have $\alpha_{(ih)} = \alpha_1$ for $i = 0, \ldots, \tau - 1$, and $\alpha_{(jh)} = \alpha_2$ for $j = \tau, \ldots, k$. The prove of multiple switching case can be obtained by simple analogy to the proof of one switching case. ■

3.2 Experimental setup

Analog model of the $\mathcal{A}$-type operator realized directly based on its switching order scheme is depicted in Fig. 3.3. The order strictly depends on fractional order impedances Z_1 and Z_2. The half-order impedances are designed according to the algorithm shown in Section 1.6.1. Additionally, the analog model contains such elements as: operational amplifiers TL071, analog switches DG303 and resistors R. The amplifiers A_1 and A_2 are responsible for keeping the proper order of the variable-order system. However, the output signals of the amplifiers possess the inverted polarization, therefore another proportional amplifiers A_2 and A_4 are added to the previous ones. To keep the behavior of the $\mathcal{A}$-type operator, the switches S_1 and S_2, are connected to terminals b first, and after switching time, the switches change their positions to terminals a. So, the amplifiers A_3, A_4 are being reduced.

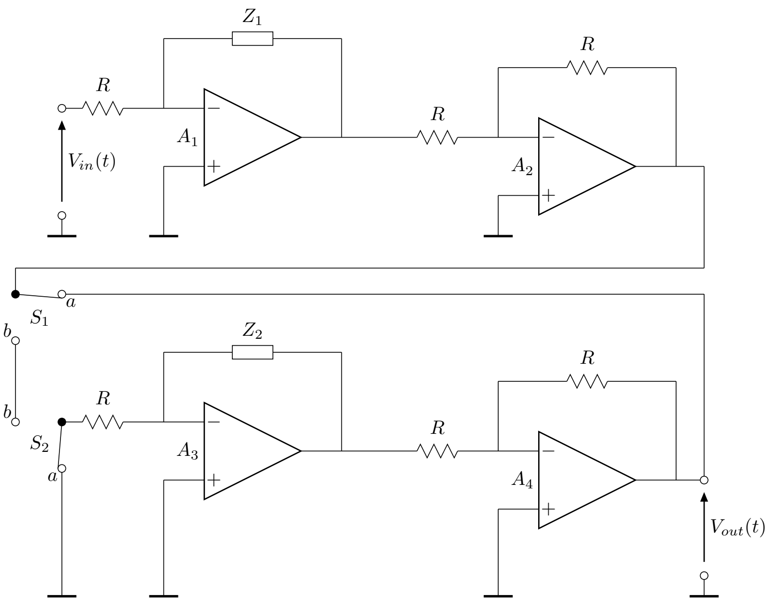


Fig. 3.3. Analog realization of the $\mathcal{A}$-type fractional variable-order integrator.

3.2.1 Results of the $\mathcal{A}$-type integrator

Example 3.5. [31] Integrator with order switching from $\alpha = 0.5$ to $\alpha = 1$.

In this case, the structure of analog model presented in Fig. 3.3 takes the following values: Z_1 and Z_2 are the half-order impedances, resistors $R = 47\mathrm{k}\Omega$. The identification results are obtained by numerical minimization of time responses square error with sampling time 0.001 s and input signal being the scalled Heaviside step function $u(t) = 0.1 \cdot 1(t)\mathrm{V}$.

After identification process the following analog models of half- and first-order integrators are obtained, respectively:

$$\begin{aligned} y(t) &= {}_0\mathrm{D}_t^{-0.5} a_1 u(t) = 2.8 {}_0\mathrm{D}_t^{-0.5} u(t), \\ y(t) &= {}_0\mathrm{D}_t^{-1} a_2 u(t) = 8.1 {}_0\mathrm{D}_t^{-1} u(t), \end{aligned}$$

which gives rise to the following variable-order integrator

$$y(t) = {}^{\mathcal{A}}_0\mathrm{D}_t^{-\alpha(t)} \left[a(t) u(t) \right],$$

where (for switching time $T = 0.3$ s)

$$a(t) = \begin{cases} 8.1 & \text{for } t < 0.3, \\ 2.8 & \text{for } t \geq 0.3, \end{cases}$$

and

$$\alpha(t) = \begin{cases} 1 & \text{for } t < 0.3, \\ 0.5 & \text{for } t \geq 0.3. \end{cases}$$

Identification results, and difference between step responses of the half and quarter-order integrators are presented in Fig. 3.4 and Fig. 3.5, respectively.

The experimental results compared to numerical implementation of the $\mathcal{A}$-type variable-order integrator are presented in Fig. 3.6.

3.3 Summary

In this chapter, introduction to the $\mathcal{A}$-type fractional variable-order derivative was presented. In this part, the matrix form corresponding to such definition was given. Then, the interpretation of the $\mathcal{A}$-type operator was presented as a switching order scheme. The second part is fully devoted to experimental setup. In this part, the analog model being the realization of the $\mathcal{A}$-type derivative was given and used to gather the experimental data. At the end, the experimental data were compared with their numerical implementations.

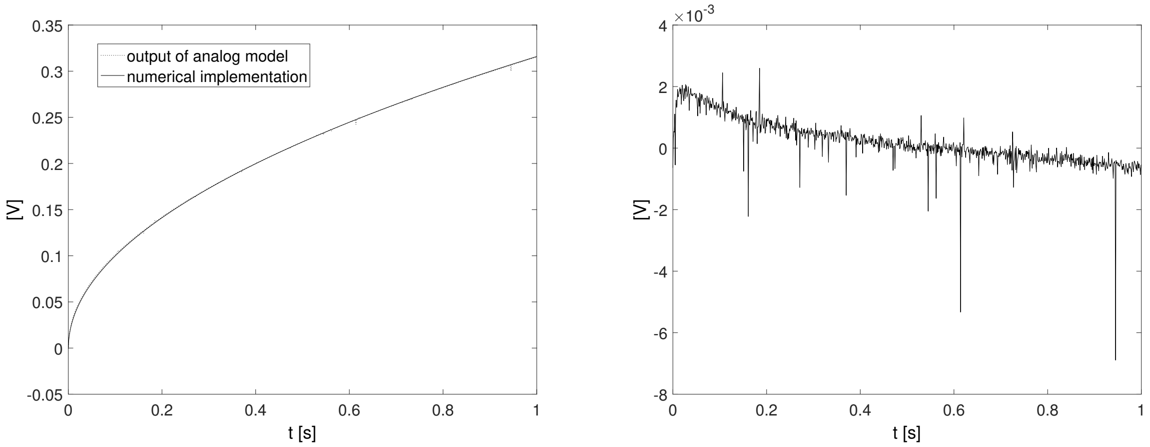


Fig. 3.4. Identification results of constant-order $\alpha = 0.5$ integrator (*left*) and their modeling error (*right*).

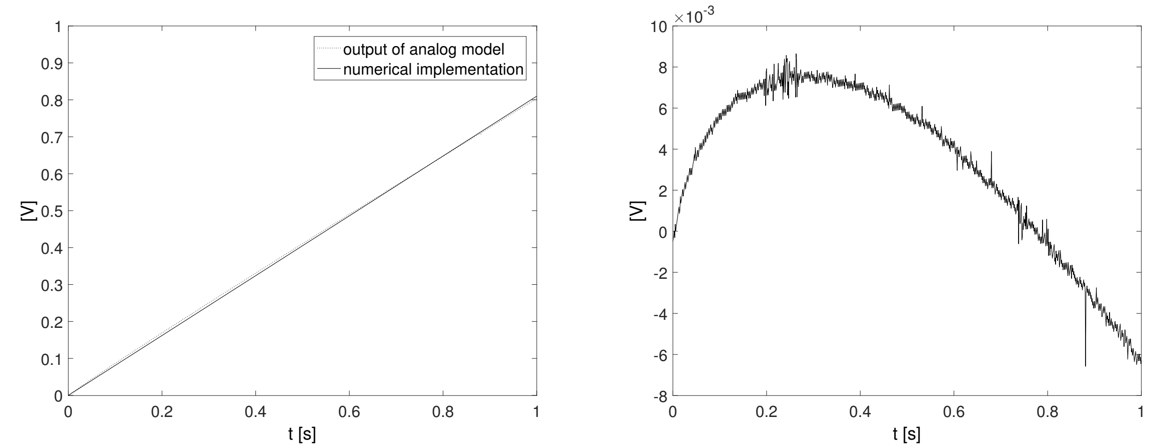


Fig. 3.5. Identification results of constant-order $\alpha = 1$ integrator (*left*) and their modeling error (*right*).

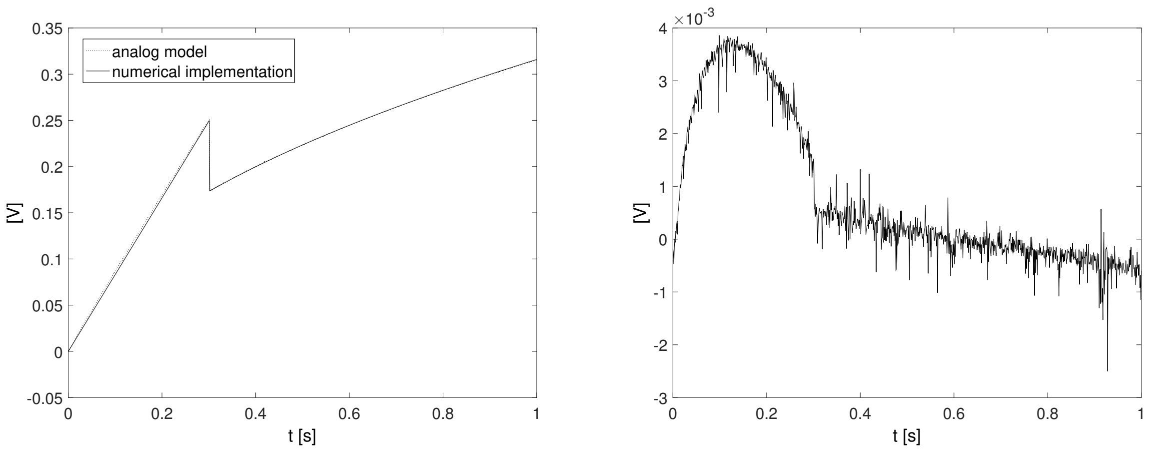


Fig. 3.6. Results of analog and numerical implementation of the $\mathcal{A}$-type variable-order integrator with order switching from $\alpha = 0.5$ to $\alpha = 1$ (*left*) and their modeling error (*right*).

Chapter 4

The $\mathcal{B}$-type fractional variable-order definition

In this chapter, the $\mathcal{B}$-type fractional variable-order difference and derivative are presented together with their matrix forms. The simple and multi-switching schemes equivalent to $\mathcal{B}$-type definition are introduced. Moreover, to validate the theoretical considerations, the analog model being the realization of such switching scheme is built and used to gather the experimental data of $\mathcal{B}$-type integrator and $\mathcal{B}$-type inertial systems. At the end, the experimental data are compared to their numerical implementations.

4.1 Introduction to $\mathcal{B}$-type operator

In the $\mathcal{B}$-type operator, the arguments of order and input function are defined in the same way, it means that coefficients for past samples are calculated with order which was present for these samples. In this case, the definition has the following form:

Definition 4.1. [26, 104] The $\mathcal{B}$-type of fractional variable-order derivative is as follows

$$ {}_{0}^{\mathcal{B}}\mathrm{D}_{t}^{\alpha(t)} f(t) = \lim_{h\to 0} \sum_{j=0}^{n} \frac{(-1)^j}{h^{\alpha(t-jh)}} \binom{\alpha(t-jh)}{j} f(t-jh), $$

where $n = \lfloor t/h \rfloor$ and h is a sampling time.

Definition 4.2. [26, 104] The discrete realization of the $\mathcal{B}$-type fractional variable-order derivative is as follows

$$ {}_{0}^{\mathcal{B}}\Delta_{l}^{\alpha_l} f_l = \sum_{j=0}^{k} \frac{(-1)^j}{h^{\alpha_{l-j}}} \binom{\alpha_{l-j}}{j} f_{l-j}, $$

for $l = 0, 1, 2, \ldots, k$.

In Fig. 4.1, plots of step function derivatives (according to Def. 4.1) are presented for

$\alpha_1 = -1$, $\alpha_2 = -2$, and

$$\alpha_3(t) = \begin{cases} -1 & \text{for } 0 \leq t < 1, \\ -2 & \text{for } 1 \leq t \leq 2. \end{cases} \tag{4.1}$$

In case of the $\mathcal{B}$-type derivative, at the switching time instant, the output of derivative

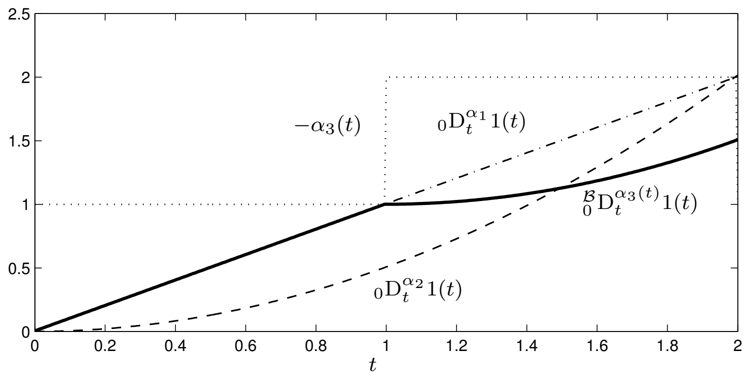


Fig. 4.1. Plots of step function derivatives with respect to the $\mathcal{B}$-type derivative (given by Def. 4.1).

does not overlap the plot for constant-order α_1 and starts to integrate like at the beginning of the plot for constant-order α_2, but with some initial value.

The interesting order composition properties of the $\mathcal{B}$-type definition with constant-order operator are proved in [34]. It tourned out that the order composition of $\mathcal{B}$-type operator with constant-order differ-integral is not commutative and holds only in one direction:

Theorem 4.3. *[34] The order composition properties of $\mathcal{B}$-type operator with fractional constant-order operator is given as follows*

$${}_0\mathrm{D}_t^{\beta}\left({}_0^{\mathcal{B}}\mathrm{D}_t^{\alpha(t)}f(t)\right) = {}_0^{\mathcal{B}}\mathrm{D}_t^{\alpha(t)+\beta}f(t)$$

$${}_0^{\mathcal{B}}\mathrm{D}_t^{\alpha(t)}{}_0\left(\mathrm{D}_t^{\beta}f(t)\right) \neq {}_0^{\mathcal{B}}\mathrm{D}_t^{\alpha(t)+\beta}f(t),$$

where $\alpha(t) \neq const$ and $\beta \neq 0$.

4.2 Matrix form of the $\mathcal{B}$-type operator

By extension of Definition 4.1 and Definition 4.2 there is possibility to achieve their matrix representations. The matrix form of the $\mathcal{B}$-type fractional variable-order difference is given by [86]

$$\begin{pmatrix} {}^{\mathcal{B}}_{0}\Delta^{\alpha_0}_{0} f_0 \\ {}^{\mathcal{B}}_{0}\Delta^{\alpha_1}_{1} f_1 \\ {}^{\mathcal{B}}_{0}\Delta^{\alpha_2}_{2} f_2 \\ {}^{\mathcal{B}}_{0}\Delta^{\alpha_3}_{3} f_3 \\ \vdots \\ {}^{\mathcal{B}}_{0}\Delta^{\alpha_k}_{k} f_k \end{pmatrix} = \begin{pmatrix} h^{-\alpha_0} & 0 & 0 & \dots & 0 \\ w_{\alpha_1,1} & h^{-\alpha_1} & 0 & \dots & 0 \\ w_{\alpha_1,2} & w_{\alpha_2,1} & h^{-\alpha_2} & \dots & 0 \\ w_{\alpha_1,3} & w_{\alpha_2,2} & w_{\alpha_3,1} & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ w_{\alpha_1,k} & w_{\alpha_2,k-1} & w_{\alpha_3,k-2} & \dots & h^{-\alpha_k} \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_k \end{pmatrix},$$

where $w_{\alpha,i} = \frac{(-1)^i \binom{\alpha}{i}}{h^\alpha}$, and $h = t/k$ is a time step, k is the number of samples. Taking the limit $h \to 0$ the following matrix form of $\mathcal{B}$-type derivative is given [86]:

$$\begin{pmatrix} {}^{\mathcal{B}}_{0}\mathrm{D}^{\alpha(t)}_{0} f(0) \\ {}^{\mathcal{B}}_{0}\mathrm{D}^{\alpha(t)}_{h} f(h) \\ {}^{\mathcal{B}}_{0}\mathrm{D}^{\alpha(t)}_{2h} f(2h) \\ \vdots \\ {}^{\mathcal{B}}_{0}\mathrm{D}^{\alpha(t)}_{kh} f(kh) \end{pmatrix} = \lim_{h\to 0} {}^{\mathcal{B}}W(\alpha, k) \begin{pmatrix} f(0) \\ f(h) \\ f(2h) \\ \vdots \\ f(kh) \end{pmatrix},$$

where

$${}^{\mathcal{B}}W(\alpha, k) = \begin{pmatrix} h^{-\alpha(0)} & 0 & 0 & \dots & 0 \\ w_{\alpha(h),1} & h^{-\alpha(h)} & 0 & \dots & 0 \\ w_{\alpha(h),2} & w_{\alpha(2h),1} & h^{-\alpha(2h)} & \dots & 0 \\ w_{\alpha(h),3} & w_{\alpha(2h),2} & w_{\alpha(3h),1} & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ w_{\alpha(h),k} & w_{\alpha(2h),k-1} & w_{\alpha(3h),k-2} & \dots & h^{-\alpha(kh)} \end{pmatrix}.$$

and ${}^{B}W(\alpha, k) \in \mathbb{R}^{(k+1)\times(k+1)}$.

4.3 Numerical scheme for the $\mathcal{B}$-type operator

This section contains the block diagrams and corresponding to them numerical schemes equivalent to the $\mathcal{B}$-type operator. For better understanding, the simple-switching case between two real arbitrary constant-orders, e.g., α_1 and α_2 is introduced first. Next, this idea is generalized for a multiple-switching (variable-order) case.

Simple-switching (variable-order) case

The idea is depicted in Fig. 4.2, where all switches S_i, $i = 1, 2$, change their positions depending on an actual value of $\alpha(t)$. To switch the system from α_1 to α_2 the switches take the following terminals: until switching time, switch S_1 is connected to terminal b and switch S_2 is connected to terminal a. After this time: $S_1 = a$ and $S_2 = b$. At the instant time T, the derivative block of complementary order $\bar{\alpha}_2$ is pre-connected to the front of the current derivative block of order α_1, where

$$\bar{\alpha}_2 = \alpha_2 - \alpha_1. \tag{4.2}$$

If $\bar{\alpha}_2 < 0$, then ${}_T\mathrm{D}_t^{\bar{\alpha}_2}$ corresponds to integration of $f(t)$; and, if $\bar{\alpha}_2 > 0$, then ${}_T\mathrm{D}_t^{\bar{\alpha}_2}$ corresponds to derivative of $f(t)$, with appropriate order $\bar{\alpha}_2$.

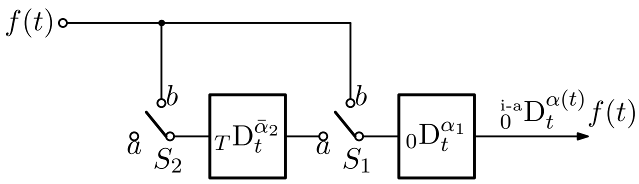


Fig. 4.2. Structure of input-additive order derivative (switching from α_1 to α_2)

Now, the numerical scheme corresponding to the input-additive switching structure (i-a) is introduced.

Lemma 4.4. *[86] For a switching order case, when the switch from order α_1 to order α_2 occurs at time T (the corresponding sample number $\hat{k}$ is such that $T = \hat{k}h$), the numerical scheme has the following form:*

$$\begin{pmatrix} {}_0^{\text{i-a}}\mathrm{D}_0^{\alpha(t)} f(t) \\ {}_0^{\text{i-a}}\mathrm{D}_h^{\alpha(t)} f(t) \\ \vdots \\ {}_0^{\text{i-a}}\mathrm{D}_{(\hat{k}-1)h}^{\alpha(t)} f(t) \\ {}_0^{\text{i-a}}\mathrm{D}_T^{\alpha(t)} f(t) \\ \vdots \\ {}_0^{\text{i-a}}\mathrm{D}_{kh}^{\alpha(t)} f(t) \end{pmatrix} = \lim_{h \to 0} W(\alpha_1, k) W(\bar{\alpha}_2, k, \hat{k}) \begin{pmatrix} f(0) \\ f(h) \\ \vdots \\ f(\hat{k}h - h) \\ f(T) \\ \vdots \\ f(kh) \end{pmatrix}, \tag{4.3}$$

where

$$W(\bar{\alpha}_2, k, \hat{k}) = \begin{pmatrix} I_{\hat{k},\hat{k}} & 0_{\hat{k},k-\hat{k}+1} \\ 0_{k-\hat{k}+1,\hat{k}} & W(\bar{\alpha}_2, k - \hat{k}) \end{pmatrix},$$

and

$$\alpha(t) = \begin{cases} \alpha_1 & \text{for } t < T, \\ \alpha_1 + \bar{\alpha}_2 & \text{for } t \geq T. \end{cases}$$

The order $\bar{\alpha}_2$, appearing above, is given by relation (4.2).

Proof. The signal incoming to the block of derivative α_1 can be described as follows

$$\begin{pmatrix} f(0) \\ f(h) \\ \vdots \\ f(\hat{k}h-h) \\ {}_{T}\mathrm{D}_{T}^{\bar{\alpha}_2} f(t) \\ \vdots \\ {}_{T}\mathrm{D}_{kh}^{\bar{\alpha}_2} f(t) \end{pmatrix} = \lim_{h\to 0} W(\bar{\alpha}_2, k, \hat{k}) \begin{pmatrix} f(0) \\ f(h) \\ \vdots \\ f(\hat{k}h-h) \\ f(T) \\ \vdots \\ f(kh) \end{pmatrix}. \tag{4.4}$$

Until time T, the input of α_1-block obtains the original function $f(t)$, so in matrix $W(\bar{\alpha}_2, k, \hat{k})$ an identity matrix occurs. From time step T, the input signal passes through the block of derivative $\bar{\alpha}_2$, and then the sub-matrix $W(\bar{\alpha}_2, k-\hat{k})$ is responsible for starting the $\bar{\alpha}_2$ derivative action from time T. That signal passes to the α_1-block and has the following matrix form:

$$\begin{pmatrix} {}_{0}^{\text{i-a}}\mathrm{D}_{0}^{\alpha(t)} f(t) \\ {}_{0}^{\text{i-a}}\mathrm{D}_{h}^{\alpha(t)} f(t) \\ \vdots \\ {}_{0}^{\text{i-a}}\mathrm{D}_{(\hat{k}-1)h}^{\alpha(t)} f(t) \\ {}_{0}^{\text{i-a}}\mathrm{D}_{T}^{\alpha(t)} f(t) \\ \vdots \\ {}_{0}^{\text{i-a}}\mathrm{D}_{kh}^{\alpha(t)} f(t) \end{pmatrix} = \lim_{h\to 0} W(\alpha_1, k) \begin{pmatrix} f(0) \\ f(h) \\ \vdots \\ f(\hat{k}h-h) \\ {}_{T}\mathrm{D}_{T}^{\bar{\alpha}_2} f(t) \\ \vdots \\ {}_{T}\mathrm{D}_{kh}^{\bar{\alpha}_2} f(t) \end{pmatrix}. \tag{4.5}$$

After substituting (4.4) to (4.5), we get (4.3), which ends the proof. ■

Multiple-switching (variable-order) case

In general case, when there are many switchings between arbitrary orders, we have the following structure presented in Fig. 4.3. When we switch from the order α_{j-1} to the

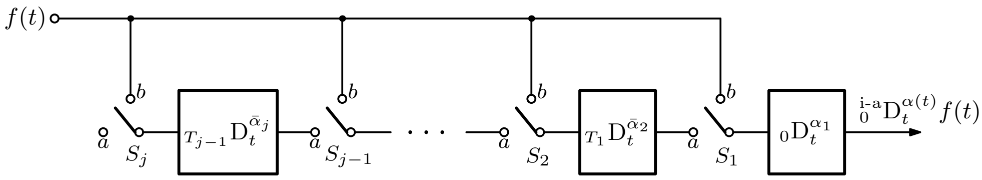


Fig. 4.3. Structure of multiple input-additive switching order derivatives

order α_j at the switch-time instant T_{j-1}, for $j = 2, 3, \ldots$, we have to set:

$$S_i = \begin{cases} a & \text{for } i = 1, \ldots, j-1, \\ b & \text{for } i = j, \end{cases}$$

and the pre-connected derivative block (on the front of the previous term) is of the following complementary order:

$$\bar{\alpha}_j = \alpha_j - \alpha_{j-1},$$

where

$$\alpha_{j-1} = \alpha_1 + \sum_{k=1}^{j-2} \bar{\alpha}_{k+1}.$$

The numerical scheme describing the already presented general case of structure allowing to switch between an arbitrary number of orders is given below.

Lemma 4.5. *[86] Matrix form of multiple input-additive switching order scheme presented in Fig. 4.3 is expressed by*

$$\begin{pmatrix} {}^{\text{i-a}}_{0}\mathrm{D}^{\alpha(t)}_{0} f(t) \\ {}^{\text{i-a}}_{0}\mathrm{D}^{\alpha(t)}_{h} f(t) \\ {}^{\text{i-a}}_{0}\mathrm{D}^{\alpha(t)}_{2h} f(t) \\ {}^{\text{i-a}}_{0}\mathrm{D}^{\alpha(t)}_{3h} f(t) \\ \vdots \\ {}^{\text{i-a}}_{0}\mathrm{D}^{\alpha(t)}_{kh} f(t) \end{pmatrix} = \begin{pmatrix} h^{-\bar{\alpha}_0} & 0 & 0 & \dots & 0 \\ w_{\bar{\alpha}_0,1} & h^{-(\bar{\alpha}_0+\bar{\alpha}_1)} & 0 & \dots & 0 \\ w_{\bar{\alpha}_0,2} & w_{\bar{\alpha}_0+\bar{\alpha}_1,1} & h^{-(\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2)} & \dots & 0 \\ w_{\bar{\alpha}_0,3} & w_{\bar{\alpha}_0+\bar{\alpha}_1,2} & w_{\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2,1} & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ w_{\bar{\alpha}_0,k} & w_{\bar{\alpha}_0+\bar{\alpha}_1,k-1} & w_{\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2,k-2} & \dots & h^{-(\bar{\alpha}_0+\dots+\bar{\alpha}_k)} \end{pmatrix} \begin{pmatrix} f(0) \\ f(h) \\ f(2h) \\ f(3h) \\ \vdots \\ f(kh) \end{pmatrix}.$$

In this description, we assume that the order of derivative changes with every time step (which gives a variable-order derivative), i.e.,

$$\alpha(t) = \alpha_j \quad \textit{for} \quad jh \leq t \leq (j+1)h, \quad j = 0, \dots, k,$$

where

$$\alpha_j = \sum_{i=0}^{j} \bar{\alpha}_i, \tag{4.6}$$

where, in this case, $\alpha_0 = \bar{\alpha}_0$ is a value of initial order.

Proof. Using Lemma 4.4, the following numerical scheme is obtained

$$\begin{pmatrix} {}^{\text{i-a}}_{0}\mathrm{D}^{\alpha(t)}_{0} f(t) \\ \vdots \\ {}^{\text{i-a}}_{0}\mathrm{D}^{\alpha(t)}_{kh} f(t) \end{pmatrix} = \prod_{j=0}^{k} W(\bar{\alpha}_j, k, j) \begin{pmatrix} f(0) \\ \vdots \\ f(kh) \end{pmatrix}.$$

The first switching matrices can be described as the following block matrices:

$$\begin{aligned} W(\bar{\alpha}_0, k, 0) W(\bar{\alpha}_1, k, 1) &= \left(\begin{array}{c|c} h^{-\bar{\alpha}_0} & 0_{1,k} \\ \hline R(\bar{\alpha}_0, 1) & W(\bar{\alpha}_0, k-1) \end{array}\right) \left(\begin{array}{c|c} 1 & 0_{1,k} \\ \hline 0 & W(\bar{\alpha}_1, k-1) \end{array}\right) \\ &= \left(\begin{array}{c|c} h^{-\bar{\alpha}_0} & 0_{1,k} \\ \hline R(\bar{\alpha}_0, 1) & W(\bar{\alpha}_0 + \bar{\alpha}_1, k-1) \end{array}\right), \end{aligned}$$

where

$$R(\bar{\alpha}, i) = \begin{pmatrix} w_{\bar{\alpha},i} \\ w_{\bar{\alpha},i+1} \\ \vdots \end{pmatrix}.$$

For a switching in the next sample time, we obtain the following numerical scheme:

$$\begin{aligned}
&W(\bar{\alpha}_0, k, 0)W(\bar{\alpha}_1, k, 1)W(\bar{\alpha}_2, k, 2) \\
&= \left(\begin{array}{cc|c} h^{-\bar{\alpha}_0} & 0 & 0_{1,k-1} \\ w_{\bar{\alpha}_0,1} & h^{-(\bar{\alpha}_0+\bar{\alpha}_1)} & 0_{1,k-1} \\ \hline R(\bar{\alpha}_0, 2) & R(\bar{\alpha}_0 + \bar{\alpha}_1, 1) & W(\bar{\alpha}_0 + \bar{\alpha}_1, k-2) \end{array}\right) \left(\begin{array}{cc|c} 1 & 0 & 0_{1,k-1} \\ 0 & 1 & 0_{1,k-1} \\ \hline 0 & 0 & W(\bar{\alpha}_2, k-1) \end{array}\right) \\
&= \left(\begin{array}{cc|c} h^{-\bar{\alpha}_0} & 0 & 0_{1,k-1} \\ w_{\bar{\alpha}_0,1} & h^{-(\bar{\alpha}_0+\bar{\alpha}_1)} & 0_{1,k-1} \\ \hline R(\bar{\alpha}_0, 2) & R(\bar{\alpha}_0 + \bar{\alpha}_1, 1) & W(\bar{\alpha}_0 + \bar{\alpha}_1 + \bar{\alpha}_2, k-2) \end{array}\right).
\end{aligned}$$

In the case of k-th switching of order, we have the following form of the switching matrix $W(\bar{\alpha}_0, k, 0)W(\bar{\alpha}_1, k, 1)\cdots W(\bar{\alpha}_k, k, k)$, i.e.,

$$\prod_{j=0}^{k} W(\bar{\alpha}_j, k, j) = \begin{pmatrix} h^{-\bar{\alpha}_0} & 0 & 0 & \dots & 0 \\ w_{\bar{\alpha}_0,1} & h^{-(\bar{\alpha}_0+\bar{\alpha}_1)} & 0 & \dots & 0 \\ w_{\bar{\alpha}_0,2} & w_{\bar{\alpha}_0+\bar{\alpha}_1,1} & h^{-(\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2)} & \dots & 0 \\ w_{\bar{\alpha}_0,3} & w_{\bar{\alpha}_0+\bar{\alpha}_1,2} & w_{\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2,1} & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ w_{\bar{\alpha}_0,k} & w_{\bar{\alpha}_0+\bar{\alpha}_1,k-1} & w_{\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2,k-2} & \dots & h^{-(\bar{\alpha}_0+\cdots+\bar{\alpha}_k)} \end{pmatrix}$$

or shortly, using the relationship (4.6)

$$\prod_{j=0}^{k} W(\bar{\alpha}_j, k, j) = \begin{pmatrix} h^{-\alpha_0} & 0 & 0 & \dots & 0 & 0 \\ w_{\alpha_0,1} & h^{-\alpha_1} & 0 & \dots & 0 & 0 \\ w_{\alpha_0,2} & w_{\alpha_1 1} & h^{-\alpha_2} & \dots & 0 & 0 \\ w_{\alpha_0,3} & w_{\alpha_1,2} & w_{\alpha_2,1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ w_{\alpha_0,k-1} & w_{\alpha_1,k-2} & w_{\alpha_2,k-3} & \dots & h^{-\alpha_{k-1}} & 0 \\ w_{\alpha_0,k} & w_{\alpha_1,k-1} & w_{\alpha_2,k-2} & \dots & w_{\alpha_{k-1},1} & h^{-\alpha_k} \end{pmatrix}. \tag{4.7}$$

■

Corollary 4.6. *Lemma 4.5 provides the equivalence between matrix form corresponding to the multiple input-additive switching order scheme (given by 4.5) and the* $\mathcal{B}$*-type variable-order derivative.*

4.4 Experimental results

An analog model of switching system, used in experimental setup, directly based on the switching scheme given in Fig. 4.3 is shown in Fig. 4.4. The experimental setup is prepared according to simple-switching block diagram. The first-order integrator is realized according to traditional scheme based on capacitor while structure of fractional order impedances are built according to half- and quarter-order approximations (domino-ladder structures). Due to the inverted signals at the output of operational amplifier, the proportional gains -1 for integrators A_1 and A_3 are added. In general cases, the scheme based on amplifiers A_1 and A_3 contains resistor R and impedances Z_1, Z_2, Z_3 chosen according to the particular value of orders. As a realization of switches S_1 and S_2, integrated analog switches DG303 are used. The experimental circuit is connected to data acquision card dSPACE DS1104.

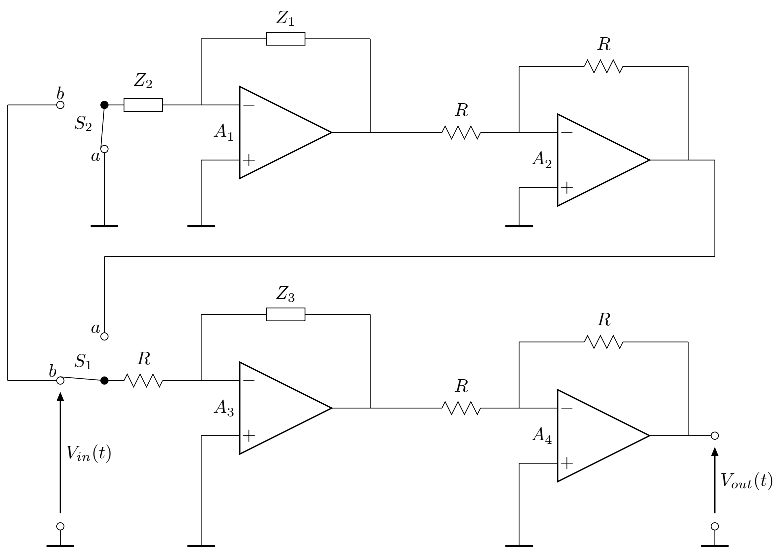


Fig. 4.4. Analog realization of the $\mathcal{B}$-type integrator.

4.4.1 Increasing of integration order

Example 4.7. [82] Integrator with order switching from $\alpha = 0.25$ to $\alpha = 0.5$.

In this configuration, the scheme in Fig. 4.4 takes the following structure: Z_1 and Z_3 are the quarter-order integrators, $Z_2 = R = 100\text{k}\Omega$. The identification results are obtained by numerical minimization of time responses square error with sampling time $h = 0.001$ s and input signal being $u(t) = 0.01 \cdot 1(t)$ V.

After identification process, the following models for orders -0.25 and -0.5 are obtained, respectively:

$$y(t) = {}_0\mathrm{D}_t^{-0.25} a_1 u(t) = 23.93 {}_0\mathrm{D}_t^{-0.25} u(t),$$
$$y(t) = {}_0\mathrm{D}_t^{-0.5} a_2 u(t) = 5.79 {}_0\mathrm{D}_t^{-0.5} u(t),$$

which gives rise to the following variable-order integrator:

$$y(t) = {}^{\mathcal{B}}_{0}\mathrm{D}_t^{-\alpha(t)} \left[a(t)u(t)\right],$$

where (for the switching time $T = 0.7$ s)

$$a(t) = \begin{cases} 23.93 & \text{for } t < 0.7, \\ 5.79 & \text{for } t \geq 0.7, \end{cases}$$

and

$$\alpha(t) = \begin{cases} 0.25 & \text{for } t < 0.7, \\ 0.5 & \text{for } t \geq 0.7. \end{cases}$$

Identification results of quarter- and half-order integrators together with their modeling errors are presented in Fig. 4.5 and Fig. 4.6, respectively. The experimental results of

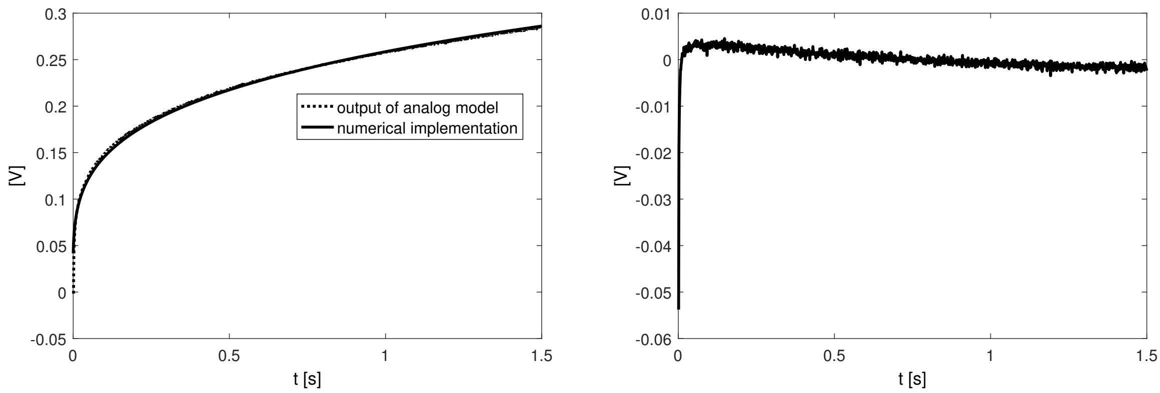


Fig. 4.5. Identification results of $\alpha = 0.25$ order integrator (*left*) and their modeling error (*right*).

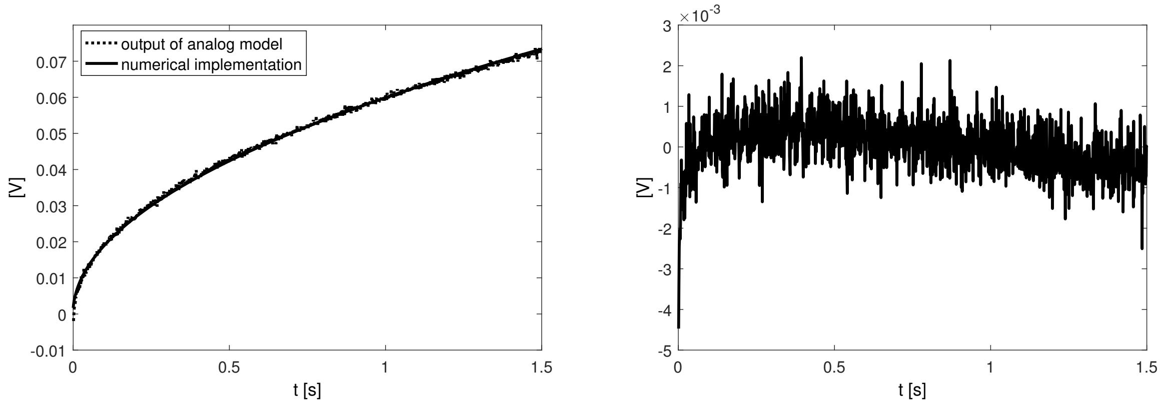


Fig. 4.6. Identification results of $\alpha = 0.5$ order integrator (*left*) and their modeling error (*right*).

the $\mathcal{B}$-type integrator with order switching from $\alpha = 0.25$ to $\alpha = 0.5$, compared to their numerical implementation, are presented in Fig. 4.7.

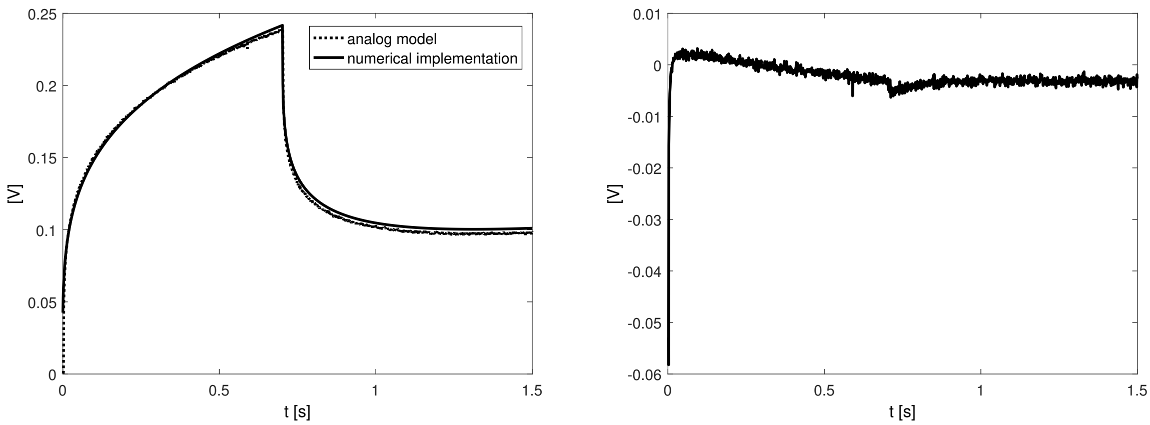


Fig. 4.7. Results of analog and numerical implementation of the $\mathcal{B}$-type variable-order integrator with order switching from $\alpha = 0.25$ to $\alpha = 0.5$ (*left*) and their modeling error (*right*).

Example 4.8. [82] Integrator with order switching from $\alpha = 0.5$ to $\alpha = 1$.

In this configuration, the scheme in Fig. 4.4 takes the following structure: Z_1 and Z_3 are the half-order impedances, $Z_2 = R = 100\text{k}\Omega$. The sampling time equals to $h = 0.001$ s and input signal being $u(t) = 0.01 \cdot 1(t)$ V. The parameters of the analog models are obtained by identification method based on step responses, separately for both orders.

After identification process, the following models for orders -0.5 and -1 are obtained, respectively:

$$y(t) = {}_0\mathrm{D}_t^{-0.5} a_1 u(t) = 17.24 {}_0\mathrm{D}_t^{-0.5} u(t),$$
$$y(t) = {}_0\mathrm{D}_t^{-1} a_2 u(t) = 24.39 {}_0\mathrm{D}_t^{-1} u(t),$$

which gives rise to the following variable-order integrator:

$$y(t) = {}_0^{\mathcal{B}}\mathrm{D}_t^{-\alpha(t)} \left[a(t) u(t) \right],$$

where (for the switching time $T = 1$ s)

$$a(t) = \begin{cases} 17.24 & \text{for } t < 1, \\ 24.39 & \text{for } t \geq 1, \end{cases}$$

and

$$\alpha(t) = \begin{cases} 0.5 & \text{for } t < 1, \\ 1 & \text{for } t \geq 1. \end{cases}$$

The experimental results of the $\mathcal{B}$-type integrator with order switching from $\alpha = 0.5$ to $\alpha = 1$, compared to their numerical implementation, are presented in Fig. 4.8.

Example 4.9. [82] Integrator with order switching form $\alpha = 0.5$ to $\alpha = 1.5$.

In this configuration, the scheme presented in Fig. 4.4 takes the following structure:

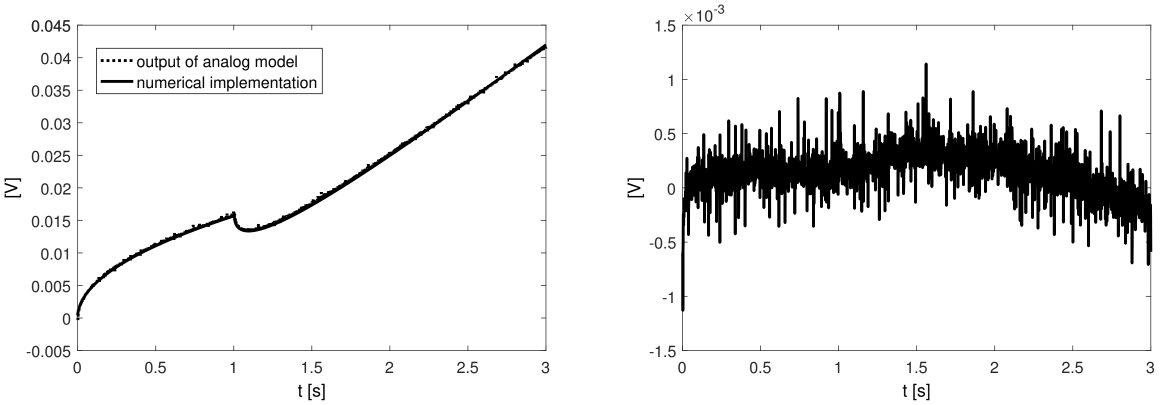


Fig. 4.8. Results of analog and numerical implementation of the $\mathcal{B}$-type integrator with order switching from $\alpha = 0.5$ to $\alpha = 1$ (*left*) and their modeling error (*right*).

Z_1 is the first-order capacitor, Z_3 is a half-order impedance and $Z_2 = R = 9.65\text{k}\Omega$. The sampling time equals to $h = 0.01$ s and input signal being $u(t) = 0.01 \cdot 1(t)$ V.

After identification process, the following models for orders -0.5 and -1.5 are obtained, respectively:

$$
\begin{aligned}
y(t) &= {}_0\mathrm{D}_t^{-0.5} a_1 u(t) = 1.47 {}_0\mathrm{D}_t^{-0.5} u(t), \\
y(t) &= {}_0\mathrm{D}_t^{-1.5} a_2 u(t) = 33.9 {}_0\mathrm{D}_t^{-1.5} u(t),
\end{aligned}
$$

which gives rise to the following variable-order integrator:

$$
y(t) = {}_0^{\mathcal{B}}\mathrm{D}_t^{-\alpha(t)} \left[a(t) u(t) \right],
$$

where (for the switching time $T = 1$ s)

$$
a(t) = \begin{cases} 1.47 & \text{for } t < 1, \\ 33.9 & \text{for } t \geq 1, \end{cases}
$$

and

$$
\alpha(t) = \begin{cases} 0.5 & \text{for } t < 1, \\ 1.5 & \text{for } t \geq 1. \end{cases}
$$

The experimental results of the $\mathcal{B}$-type integrator with order switching from $\alpha = 0.5$ to $\alpha = 1.5$, compared to their numerical implementation, are presented in Fig. 4.9.

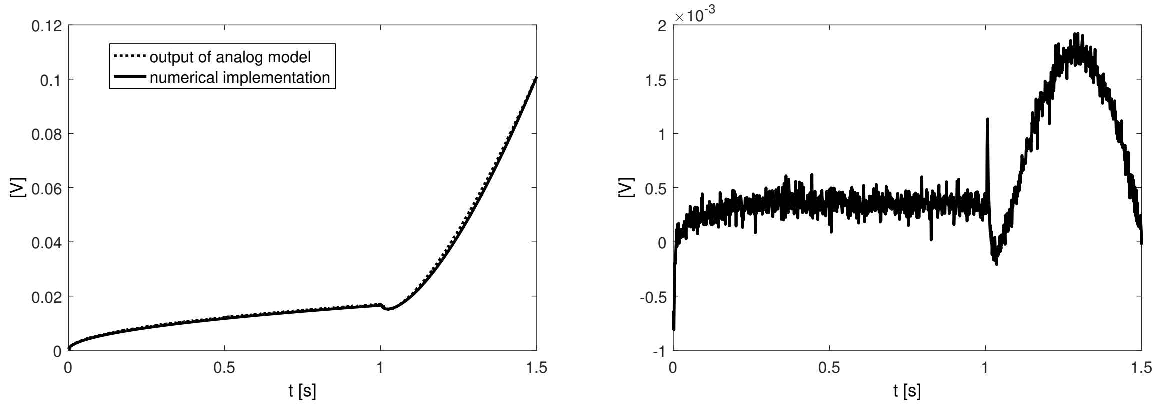


Fig. 4.9. Results of analog and numerical implementation of the $\mathcal{B}$-type integrator with order switching from $\alpha = 0.5$ to $\alpha = 1.5$ (*left*) and their modeling error (*right*)

Example 4.10. [82] Integrator with order switching from $\alpha = 1$ to $\alpha = 1.5$

In this configuration, the scheme in Fig. 4.4 takes the following structure: Z_1 is a half-order impedance, Z_3 is the first order capacitor and $Z_2 = R = 9.65\text{k}\Omega$. The sampling time equals to $h = 0.01$ s and input signal being $u(t) = 0.01 \cdot 1(t)$ V.

After identification process, the following models for orders -1 and -1.5 are obtained, respectively:

$$y(t) = {}_0\mathrm{D}_t^{-1} a_1 u(t) = 26.32 {}_0\mathrm{D}_t^{-1} u(t),$$
$$y(t) = {}_0\mathrm{D}_t^{-1.5} a_2 u(t) = 37.03 {}_0\mathrm{D}_t^{-1.5} u(t),$$

which gives rise to the following variable-order integrator:

$$y(t) = {}_0^{\mathcal{B}}\mathrm{D}_t^{-\alpha(t)} \left[a(t) u(t) \right],$$

where (for the switching time $T = 0.7$ s)

$$a(t) = \begin{cases} 26.32 & \text{for } t < 0.7, \\ 37.03 & \text{for } t \geq 0.7, \end{cases}$$

and

$$\alpha(t) = \begin{cases} 1 & \text{for } t < 0.7, \\ 1.5 & \text{for } t \geq 0.7. \end{cases}$$

The experimental results of the $\mathcal{B}$-type integrator with order switching from $\alpha = 1$ to $\alpha = 1.5$, compared to the numerical implementation, are presented in Fig. 4.10.

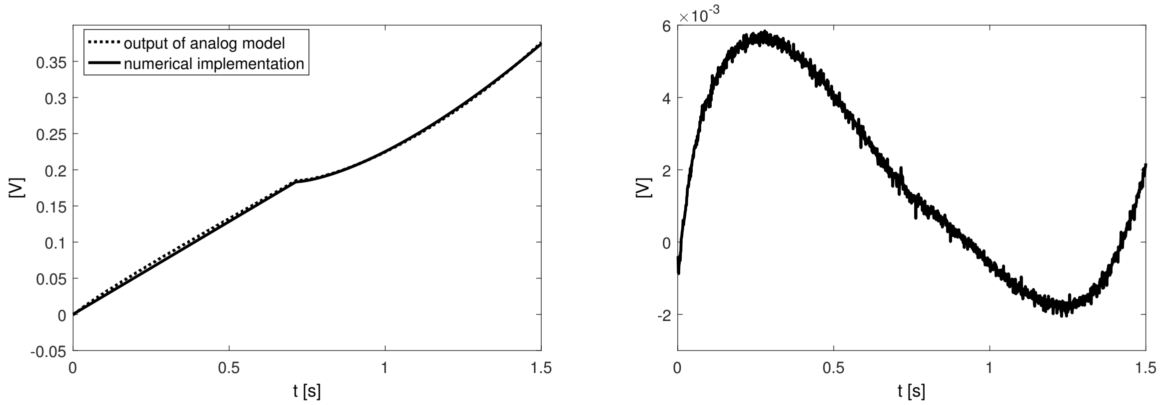


Fig. 4.10. Results of analog and numerical implementation of the $\mathcal{B}$-type integrator with order switching from $\alpha = 1$ to $\alpha = 1.5$ (*left*) and their modeling error (*right*).

4.4.2 Decreasing of integration order

Example 4.11. [82] Integrator with order switching from $\alpha = 0.5$ to $\alpha = 0.25$.

In this configuration, the scheme in Fig. 4.4 takes the following structure: $Z_1 = R = 100\text{k}\Omega$, Z_3 is a half-order impedance, and Z_2 is a quarter-order impedance.

After identification process, the following models for orders -0.5 and -0.25 are obtained, respectively:

$$y(t) = {}_0\mathrm{D}_t^{-0.5} a_1 u(t) = 6.08 {}_0\mathrm{D}_t^{-0.5} u(t),$$
$$y(t) = {}_0\mathrm{D}_t^{-0.25} a_2 u(t) = 1.47 {}_0\mathrm{D}_t^{-0.25} u(t),$$

which gives rise to the following variable-order integrator:

$$y(t) = {}^{\mathcal{B}}_0\mathrm{D}_t^{-\alpha(t)} \left[a(t)u(t)\right],$$

where (for the switching time $T = 0.7$ s)

$$a(t) = \begin{cases} 6.08 & \text{for } t < 0.7, \\ 1.47 & \text{for } t \geq 0.7, \end{cases}$$

and

$$\alpha(t) = \begin{cases} 0.5 & \text{for } t < 0.7, \\ 0.25 & \text{for } t \geq 0.7. \end{cases}$$

The experimental results of the $\mathcal{B}$-type integrator with order switching from $\alpha = 0.5$ to $\alpha = 0.25$, compared to their numerical results, are presented in Fig. 4.11. In this case, to obtain integrator of order 0.25 from integrator of order 0.5 the derivative of order 0.25 realized on amplifiers A_1 and A_2 is added to the input of the first integrator (amplifiers A_3 and A_4). Figure 4.12 presents the output signal of amplifier A_2, which is an output

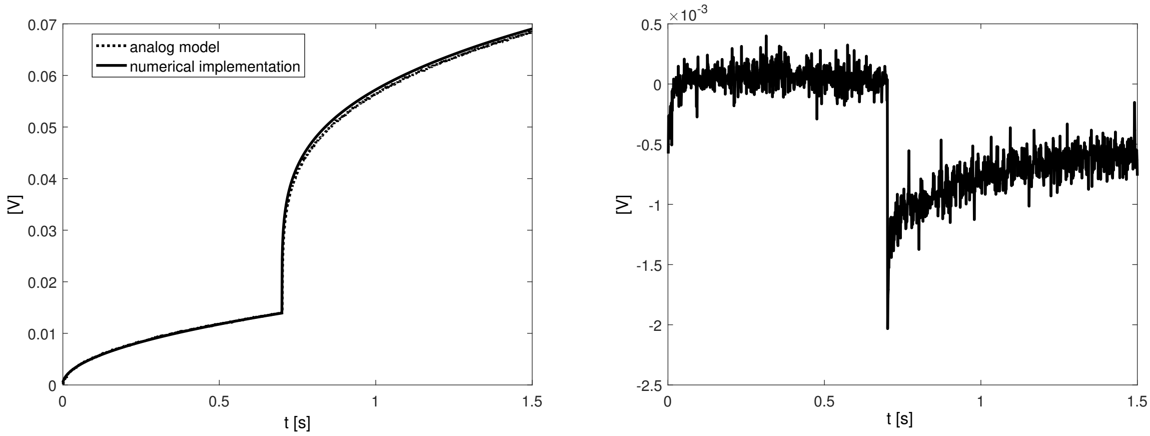


Fig. 4.11. Results of analog and numerical implementation of the $\mathcal{B}$-type integrator with order switching from $\alpha = 0.5$ to $\alpha = 0.25$ (*left*) and their modeling error (*right*).

of added derivative. As it can be noticed, the error of variable-order derivative has the same character as a modeling error of added derivative. However, presented results show high accuracy of the proposed method.

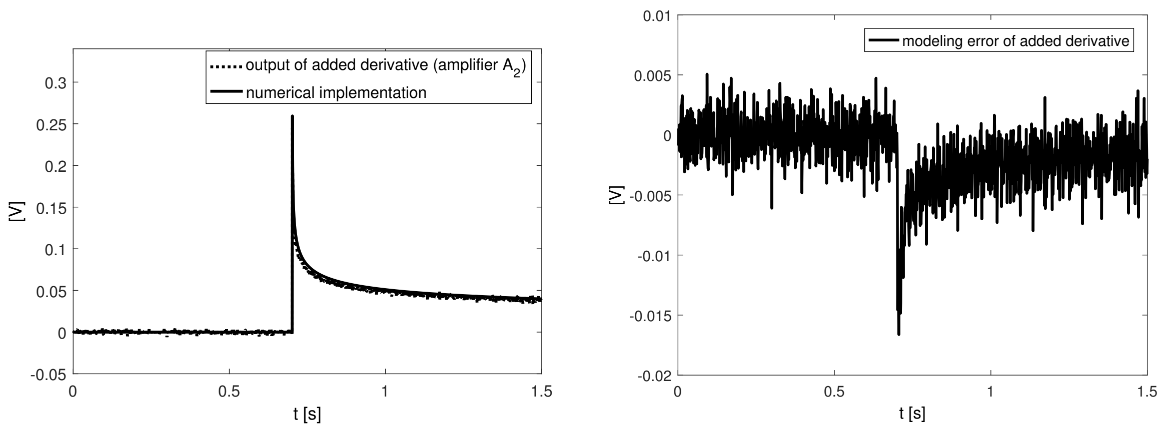


Fig. 4.12. Results of analog and numerical implementation of added derivative (output of amplifier A_2) (*left*) and modeling error (*right*).

Example 4.12. [82] Integrator with order switching from $\alpha = 1$ to $\alpha = 0.5$. In this configuration, the scheme in Fig. 4.4 takes the following structure: $Z_1 = R = 100\text{k}\Omega$, Z_2 is a half-order impedance, and finally, Z_3 is the first-order integrator based on capacitor $C = 2.2\mu\text{F}$.

After identification process, the following models for orders -1 and -0.5 are obtained, respectively:

$$y(t) = {}_0\text{D}_t^{-1} a_1 u(t) = 17.24 {}_0\text{D}_t^{-1} u(t),$$
$$y(t) = {}_0\text{D}_t^{-0.5} a_2 u(t) = 20.92 {}_0\text{D}_t^{-0.5} u(t),$$

which gives rise to the following variable-order integrator:

$$y(t) = {}_0^{\mathcal{B}}\mathrm{D}_t^{-\alpha(t)}\left[a(t)u(t)\right],$$

where (for the switching time $T = 0.7$ s)

$$a(t) = \begin{cases} 17.24 & \text{for } t < 0.7, \\ 20.92 & \text{for } t \geq 0.7, \end{cases}$$

and

$$\alpha(t) = \begin{cases} 1 & \text{for } t < 0.7, \\ 0.5 & \text{for } t \geq 0.7. \end{cases}$$

The experimental results of the $\mathcal{B}$-type integrator with order switching from $\alpha = 1$ to $\alpha = 0.5$, compared to their numerical results, are presented in Fig. 4.13. In this case, to

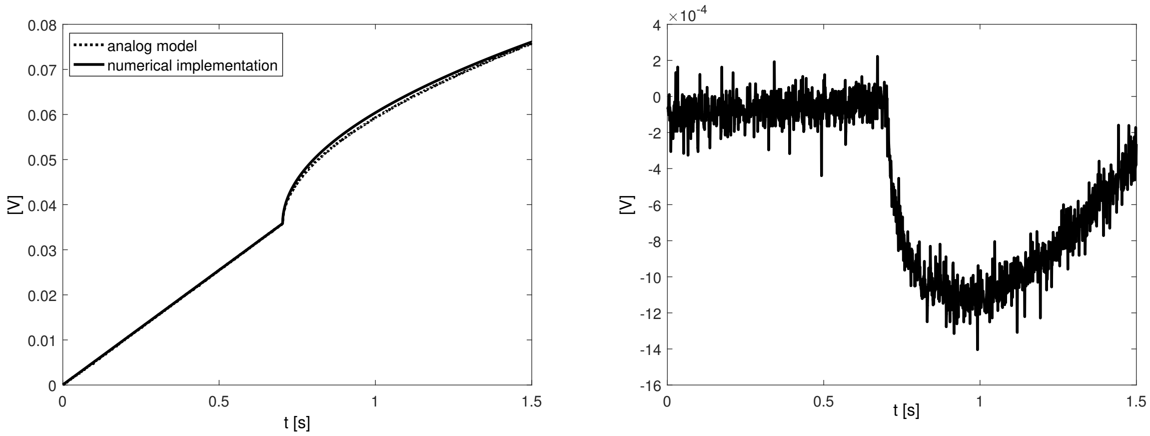


Fig. 4.13. Results of analog and numerical implementation of the $\mathcal{B}$-type integrator with order switching from $\alpha = 1$ to $\alpha = 0.5$ (*left*) and modeling error (*right*).

obtain integrator of order 0.5 from an integrator of order 1, the derivative of order 0.5 realized on amplifiers A_1 and A_2 is added to the input of the first integrator (amplifiers A_3 and A_4). Figure 4.14 presents the output signal of amplifier A_2, which is output of added derivative. As it can be noticed, the error of variable-order derivative is bigger than in the previous example and contributes to the final modeling error. This error can be caused by domino-ladder approximation, which has good accuracy only in restricted frequency range. However, the presented results show good accuracy of the proposed method.

4.4.3 Results of the $\mathcal{B}$-type inertial system

If the $\mathcal{B}$-type fractional variable-order integrator presented in Fig. 4.4 appears in forward path in unity feedback system shown in Fig. 4.15, then the $\mathcal{B}$-type fractional variable-order inertial system can be taken into account. In general, depending on variable-order

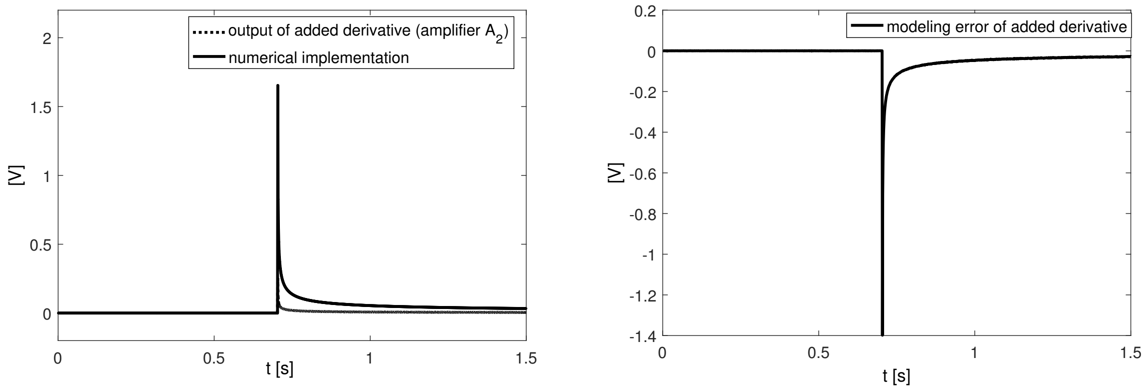


Fig. 4.14. Results of analog and numerical implementation of added derivative (output of amplifier A_2) (*left*) and modeling error (*right*)

operator type appearing in forward path, the structure of dynamical system presented in Fig. 4.15 will be so-called the fractional order inertial system.

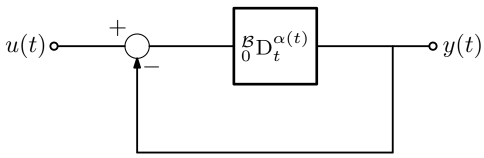


Fig. 4.15. Realization of the $\mathcal{B}$-type fractional variable-order inertial system based on fractional variable-order integral system presented in Fig. 4.4.

Example 4.13. Inertial system with order switching from -0.5 to -1.

In this case, the scheme presented in Fig. 4.4 takes the following structure: Z_1 and Z_3 are the half-order impedances and resistors $Z_2 = R = 100\mathrm{k}\Omega$. The identification results are obtained by numerical minimization of time responses square error with sampling time 0.001 s and input signal being $u(t) = 0.3 \cdot 1(t)$ V. Until switching time the S_1 switch is connected to terminal marked as b, and S_2 switch is connected to terminal a. After switching time ($T \geq 0.1s$) switches change their positions.

After identification process the following analog models for orders -0.5 and -1 are obtained, respectively:

$$y(t) = {}_0\mathrm{D}_t^{-0.5} a_1 u(t) = 1.26 {}_0\mathrm{D}_t^{-0.5} u(t),$$
$$y(t) = {}_0\mathrm{D}_t^{-1} a_2 u(t) = 1.68 {}_0\mathrm{D}_t^{-1} u(t),$$

which gives rise to the following variable-order inertial system:

$$y(t) = {}_0^{\mathcal{B}}\mathrm{D}_t^{\alpha(t)} \left[a(t)(u(t) - y(t)) \right],$$

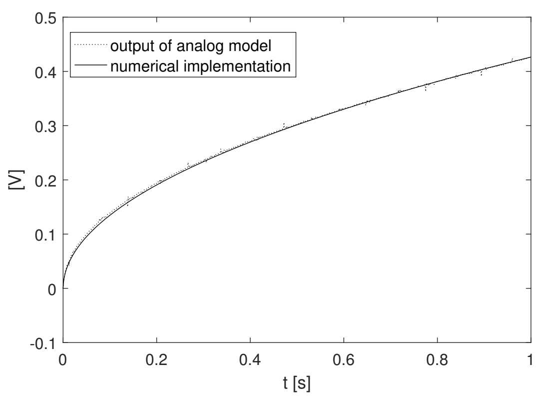
where (for the switching time $T = 0.5$ s)

$$a(t) = \begin{cases} 1.26 & \text{for } t < 0.5, \\ 1.68 & \text{for } t \geq 0.5, \end{cases}$$

and

$$\alpha(t) = \begin{cases} -0.5 & \text{for } t < 0.5, \\ -1 & \text{for } t \geq 0.5. \end{cases}$$

Identification results and modeling error of $\alpha = 0.5$ and $\alpha = 1$ integrators compared to their numerical implementations are presented in Fig. 4.16 and Fig. 4.17, respectively.



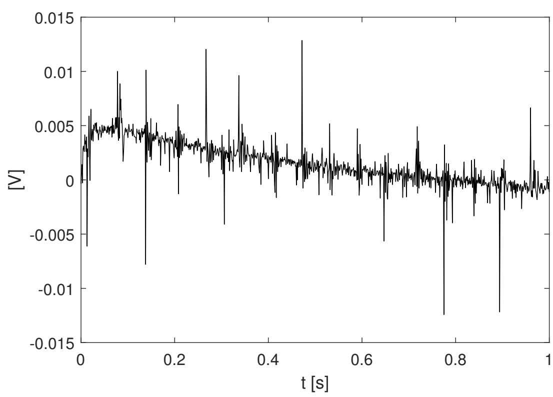


Fig. 4.16. Identification results of $\alpha = 0.5$ integrator (*left*) and their modeling error (*right*).

At the end, the experimental results compared to numerical implementation of the $\mathcal{B}$-type

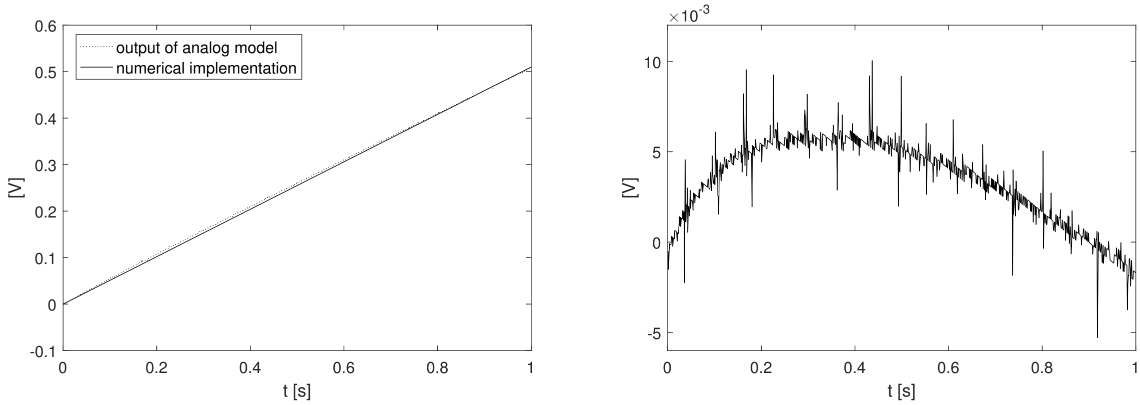


Fig. 4.17. Identification results of $\alpha = 1$ integrator (*left*) and their modeling error (*right*).

variable-order inertial system, are presented in Fig. 4.18.

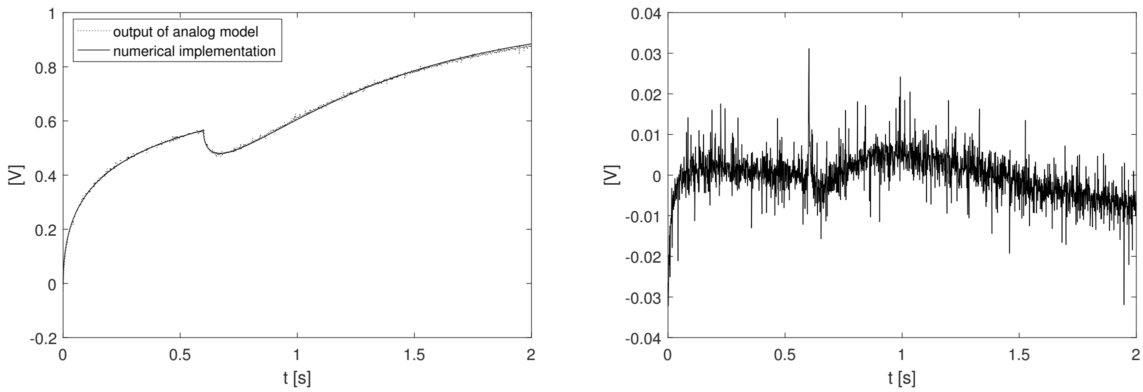


Fig. 4.18. Results of analog and numerical implementation of the $\mathcal{B}$-type inertial system with order switching from $\alpha = -0.5$ to $\alpha = -1$ (*left*) and their modeling error (*right*). The input signal equals to $1(t)$ V.

4.5 Summary

In this chapter the $\mathcal{B}$-type fractional variable-order definition was introduced with its matrix form representation. It was presented that, the behavior of $\mathcal{B}$-type operator can be interpreted as a switching order scheme. Based on it, the analog model of $\mathcal{B}$-type integrator was used to collect the experimental data for both cases: increasing and decreasing integral order. After that, the analog model of inertial system were design and compared to its numerical implementation. By extension of analog model structure the experimental setup of $\mathcal{B}$-type integral and inertial system can be easily conducted for multiple order switching.

Chapter 5

The $\mathcal{D}$-type fractional variable-order definition

In the chapter, the $\mathcal{D}$-type fractional variable-order difference and derivative are presented together with their matrix forms. The analog model corresponding to the $\mathcal{D}$-type derivative is deeply investigated in two variants: as a simple and multiple reductive-switching model. Then, the manner of dealing with initial conditions in this kind of definition is considered. At the end, two analog models are used to collect the experimental data of the $\mathcal{D}$-type integral, inertial, and inertial with initial conditions systems. Finally, experimentally obtained data are compared to their numerical implementations.

The behaviour of the $\mathcal{D}$-type operator were successfully adapted in [64] to modeling the heat transfer process in grid-holes structure.

5.1 Introduction to $\mathcal{D}$-type fractional variable-order operator

The $\mathcal{D}$-type fractional variable-order operator is related to recursive fractional constant-order definition described in Section 2.2. It can be achieved, when binomial coefficients and corresponding orders are defined in the same manner as in $\mathcal{A}$-type fractional variable-order operator. In this case, the definition has the following form:

Definition 5.1. [75, 88] The $\mathcal{D}$-type fractional variable-order derivative is as follows

$$ {}_0^{\mathcal{D}}\mathrm{D}_t^{\alpha(t)} f(t) = \lim_{h\to 0}\left(\frac{f(t)}{h^{\alpha(t)}} - \sum_{j=1}^{n}(-1)^j \binom{-\alpha(t)}{j} {}_0^{\mathcal{D}}\mathrm{D}_{t-jh}{}^{\alpha(t-jh)} f(t-jh)\right), $$

where $n = \lfloor t/h \rfloor$ and h is a sampling time.

Definition 5.2. [75, 88] The discrete realization of $\mathcal{D}$-type fractional variable-order derivative is as follows

$$ {}_0^{\mathcal{D}}\Delta_l^{\alpha_l} f_l = \frac{f_l}{h^{\alpha_l}} - \sum_{j=1}^{k}(-1)^j \binom{-\alpha_l}{j} {}_0\Delta_{l-j}^{\alpha_{l-j}} f_{l-j}, $$

for $l = 0, 1, 2, \dots, k$.

In Fig. 5.1, plots of step function derivatives (according to Def. 5.1) are presented for $\alpha_1 = -1$, $\alpha_2 = -2$, and

$$\alpha_3(t) = \begin{cases} -1 & \text{for } 0 \le t < 1, \\ -2 & \text{for } 1 \le t \le 2. \end{cases} \tag{5.1}$$

The step responses presented in Fig. 5.1 behave like a constant-order derivative until

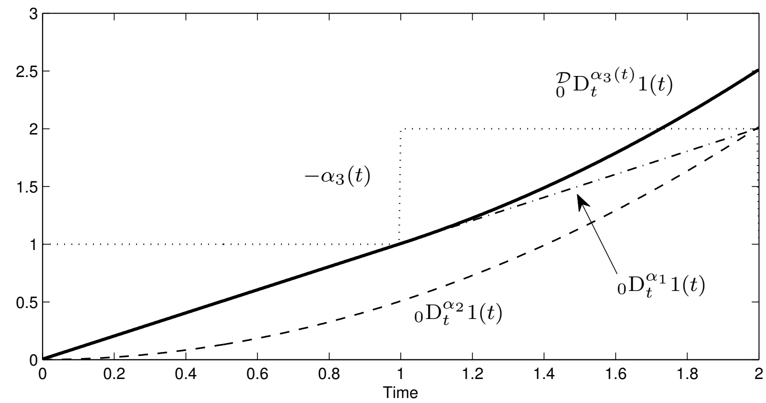


Fig. 5.1. Plots of step function derivatives with respect to the $\mathcal{D}$-type derivative (given by Def. 5.1).

switching time. After that, the output of derivative immediately changes its behavior and starts to integrate with new order α_2.

5.2 Matrix approach for the $\mathcal{D}$-type operator

By extension of Def. 5.2 there is possibility to achieve its matrix representation.

Lemma 5.3. *[88] The $\mathcal{D}$-type fractional difference given by Def. 5.2 can be expressed in the following matrix form:*

$$\begin{pmatrix} {}^{\mathcal{D}}_{0}\Delta^{\alpha_0}_{0} f_0 \\ {}^{\mathcal{D}}_{0}\Delta^{\alpha_1}_{1} f_1 \\ {}^{\mathcal{D}}_{0}\Delta^{\alpha_2}_{2} f_2 \\ \vdots \\ {}^{\mathcal{D}}_{0}\Delta^{\alpha_k}_{k} f_k \end{pmatrix} = \mathfrak{Q}^k_0 \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_k \end{pmatrix},$$

where

$$\mathfrak{Q}^k_0 = \mathfrak{Q}(\alpha_k, k) \cdots \mathfrak{Q}(\alpha_1, 1)\mathfrak{Q}(\alpha_0, 0), \tag{5.2}$$

and

$$\mathfrak{Q}(\alpha_0, 0) = \left(\begin{array}{c|c} h^{-\alpha_0} & 0_{1,k} \\ \hline 0_{k,1} & I_{k,k} \end{array}\right) \in \mathbb{R}^{(k+1)\times(k+1)},$$

and for $r = 1, \ldots, k$

$$\mathfrak{Q}(\alpha_r, r) = \left(\begin{array}{c|c|c} I_{r,r} & 0_{r,1} & 0_{r,k-r} \\ \hline q_r & h^{-\alpha_r} & 0_{1,k-r} \\ \hline 0_{k-r,r} & 0_{k-r,1} & I_{k-r,k-r} \end{array}\right) \in \mathbb{R}^{(k+1)\times(k+1)}, \tag{5.3}$$

where

$$q_r = (-w_{-\alpha_r,r}, -w_{-\alpha_r,r-1}, \ldots, -w_{-\alpha_r,1}) \in \mathbb{R}^{1\times r},$$

and $w_{-\alpha_r,i} = (-1)^i \binom{-\alpha_r}{i}$, *for* $i = 1, \ldots, r$, *i.e., the* j*th element of* q_r *is*

$$(q_r)_j = -w_{-\alpha_r,r-j+1} = (-1)^{r-j+2}\binom{-\alpha_r}{r-j+1}, \quad \textit{for} \quad j = 1, \ldots, r.$$

Proof. It is obtained after consecutive evaluating of terms from Definition (5.2) for each time step $l = 0, 1, \ldots, k$. First, for $l = 0$, we can write

$$\begin{pmatrix} {}_0^{\mathcal{D}}\Delta_0^{\alpha_0} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_k \end{pmatrix} = \begin{pmatrix} h^{-\alpha_0} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_k \end{pmatrix} = \underbrace{\begin{pmatrix} h^{-\alpha_0} & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}}_{\mathfrak{Q}(\alpha_0,0)} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_k \end{pmatrix}.$$

Next, for $l = 1$:

$$\begin{pmatrix} {}_0^{\mathcal{D}}\Delta_0^{\alpha_0} f_0 \\ {}_0^{\mathcal{D}}\Delta_1^{\alpha_1} f_1 \\ f_2 \\ \vdots \\ f_k \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ -v_{-\alpha_1,1} & h^{-\alpha_1} & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}}_{\mathfrak{Q}(\alpha_1,1)} \begin{pmatrix} {}_0^{\mathcal{D}}\Delta_0^{\alpha_0} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_k \end{pmatrix};$$

for $l = 2$:

$$\begin{pmatrix} {}^{\mathcal{D}}_{0}\Delta_0^{\alpha_0} f_0 \\ {}^{\mathcal{D}}_{0}\Delta_1^{\alpha_1} f_1 \\ {}^{\mathcal{D}}_{0}\Delta_2^{\alpha_2} f_2 \\ f_3 \\ \vdots \\ f_k \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ -v_{-\alpha_2,2} & -v_{-\alpha_2,1} & h^{-\alpha_2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}}_{\mathfrak{Q}(\alpha_2,2)} \begin{pmatrix} {}^{\mathcal{D}}_{0}\Delta_0^{\alpha_0} f_0 \\ {}^{\mathcal{D}}_{0}\Delta_1^{\alpha_1} f_1 \\ f_2 \\ \vdots \\ f_k \end{pmatrix};$$

and, generally, for $l = r$, we have

$$\begin{pmatrix} {}^{\mathcal{D}}_{0}\Delta_0^{\alpha_0} f_0 \\ {}^{\mathcal{D}}_{0}\Delta_1^{\alpha_1} f_1 \\ \vdots \\ {}^{\mathcal{D}}_{0}\Delta_{r-1}^{\alpha_{r-1}} f_{r-1} \\ {}^{\mathcal{D}}_{0}\Delta_r^{\alpha_r} f_r \\ f_{r+1} \\ \vdots \\ f_k \end{pmatrix} = \underbrace{\left(\begin{array}{c|c|c} I_{r,r} & 0_{r,1} & 0_{r,k-r} \\ \hline q_r & h^{-\alpha_r} & 0_{1,k-r} \\ \hline 0_{k-r,r} & 0_{k-r,1} & I_{k-r,k-r} \end{array}\right)}_{\mathfrak{Q}(\alpha_r,r)} \begin{pmatrix} {}^{\mathcal{D}}_{0}\Delta_0^{\alpha_0} f_0 \\ {}^{\mathcal{D}}_{0}\Delta_1^{\alpha_1} f_1 \\ \vdots \\ {}^{\mathcal{D}}_{0}\Delta_{r-1}^{\alpha_{r-1}} f_{r-1} \\ f_r \\ \vdots \\ f_k \end{pmatrix}.$$

Finally, for $l = k$:

$$\begin{pmatrix} {}^{\mathcal{D}}_{0}\Delta_0^{\alpha_0} f_0 \\ {}^{\mathcal{D}}_{0}\Delta_1^{\alpha_1} f_1 \\ \vdots \\ {}^{\mathcal{D}}_{0}\Delta_{k-1}^{\alpha_{k-1}} f_{k-1} \\ {}^{\mathcal{D}}_{0}\Delta_k^{\alpha_k} f_k \end{pmatrix} = \underbrace{\left(\begin{array}{c|c} I_{k,k} & 0_{k,1} \\ \hline q_k & h^{-\alpha_k} \end{array}\right)}_{\mathfrak{Q}(\alpha_k,k)} \begin{pmatrix} {}^{\mathcal{D}}_{0}\Delta_0^{\alpha_0} f_0 \\ {}^{\mathcal{D}}_{0}\Delta_1^{\alpha_1} f_1 \\ \vdots \\ {}^{\mathcal{D}}_{0}\Delta_{k-1}^{\alpha_{k-1}} f_{k-1} \\ f_k \end{pmatrix},$$

where $q_k = (-w_{-\alpha_k,k}, \dots, -w_{-\alpha_k,1})$.

Combining all this together, completes the proof. ■

Remark 5.4. Taking the limit $h \to 0$ to the above considerations, the following matrix form of $\mathcal{D}$-type derivative is given:

$$\begin{pmatrix} {}^{\mathcal{D}}_{0}\mathrm{D}_0^{\alpha(t)} f(0) \\ {}^{\mathcal{D}}_{0}\mathrm{D}_h^{\alpha(t)} f(h) \\ {}^{\mathcal{D}}_{0}\mathrm{D}_{2h}^{\alpha(t)} f(2h) \\ \vdots \\ {}^{\mathcal{D}}_{0}\mathrm{D}_{kh}^{\alpha(t)} f(kh) \end{pmatrix} = \lim_{h \to 0} \mathfrak{Q}_0^k \begin{pmatrix} f(0) \\ f(h) \\ f(2h) \\ \vdots \\ f(kh) \end{pmatrix}. \tag{5.4}$$

5.3 Numerical scheme for the $\mathcal{D}$-type operator

This section contains the block diagrams of simple switching case, its generalization for multiple-switching (variable-order) case and corresponding to them numerical schemes equivalent to $\mathcal{D}$-type operator.

Simple reductive-switching (variable-order) case

The reductive-switching order case occurs when the initial chain of derivatives is reduced according to changing the variable-order.

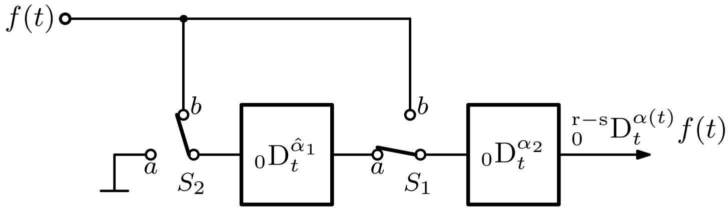


Fig. 5.2. Structure of simple reductive-switching order derivative ${}_0^{\text{r-s}}\mathrm{D}_t^{\alpha(t)} f(t)$ (switching from α_1 to α_2; configuration before time T)

The idea, introduced in [73], is depicted in Fig. 5.2, where the switches S_i, $i = 1, 2$, change their positions depending on an actual value of $\alpha(t)$. If we want to switch from α_1 to α_2, then, before switching time T, we have: $S_1 = a$, $S_2 = b$, and after this time: $S_1 = b$ and $S_2 = a$. At the instant time T, the derivative block of complementary order $\hat{\alpha}_1$ is disconnected from the front of the derivative block of order α_2, where

$$\hat{\alpha}_1 = \alpha_1 - \alpha_2. \tag{5.5}$$

Lemma 5.5. *[88] For a reductive-switching order case, when the switch from order α_1 to order α_2 occurs at time $T = \tau h$, $\tau \in \mathbb{N}_+$, the numerical scheme has the following form:*

$$\begin{pmatrix} {}_0^{\text{r-s}}\mathrm{D}_0^{\alpha(t)} f(0) \\ {}_0^{\text{r-s}}\mathrm{D}_h^{\alpha(t)} f(h) \\ \vdots \\ {}_0^{\text{r-s}}\mathrm{D}_{T-h}^{\alpha(t)} f(T-h) \\ {}_0^{\text{r-s}}\mathrm{D}_T^{\alpha(t)} f(T) \\ \vdots \\ {}_0^{\text{r-s}}\mathrm{D}_{kh}^{\alpha(t)} f(kh) \end{pmatrix} = \lim_{h\to 0} W(\alpha_2, k) W(\hat{\alpha}_1, k, \tau - 1) \begin{pmatrix} f(0) \\ f(h) \\ \vdots \\ f(T-h) \\ f(T) \\ \vdots \\ f(kh) \end{pmatrix},$$

where

$$W(\hat{\alpha}_1, k, \tau - 1) = \begin{pmatrix} W(\hat{\alpha}_1, \tau - 1) & 0_{\tau, k-\tau+1} \\ 0_{k-\tau+1, \tau} & I_{k-\tau+1, k-\tau+1} \end{pmatrix},$$

and

$$\alpha(t) = \begin{cases} \alpha_2 + \hat{\alpha}_1 & \textit{for } t < T, \\ \alpha_2 & \textit{for } t \geq T. \end{cases} \tag{5.6}$$

The order $\hat{\alpha}_1$, appearing above, is given by relation (5.5).

Proof. Until switching time T, the input of α_2-block obtains the derivative of $\hat{\alpha}_1$ signal, and beginning with time step T, the input signal of α_2-block obtain the original function $f(t)$. So, the output signal of simple reductive-switching order scheme presented in Fig. 5.2 can be described as follows

$$\begin{pmatrix} {}^{\text{r-s}}_{0}\mathrm{D}^{\alpha(t)}_{0} f(t) \\ {}^{\text{r-s}}_{0}\mathrm{D}^{\alpha(t)}_{h} f(t) \\ \vdots \\ {}^{\text{r-s}}_{0}\mathrm{D}^{\alpha(t)}_{(\hat{k}-1)h} f(t) \\ {}^{\text{r-s}}_{0}\mathrm{D}^{\alpha(t)}_{T} f(t) \\ \vdots \\ {}^{\text{r-s}}_{0}\mathrm{D}^{\alpha(t)}_{kh} f(t) \end{pmatrix} = \lim_{h\to 0} W(\bar{\alpha}_1, k, \hat{k}) \begin{pmatrix} {}_{0}\mathrm{D}^{\bar{\alpha}_1}_{0} f(0) \\ {}_{0}\mathrm{D}^{\bar{\alpha}_1}_{h} f(h) \\ \vdots \\ {}_{0}\mathrm{D}^{\bar{\alpha}_1}_{\hat{k}h-h} f(\hat{k}h - h) \\ f(T) \\ \vdots \\ f(kh) \end{pmatrix}, \tag{5.7}$$

which finally ends the proof. ∎

Multiple reductive-switching (variable-order) case

In general case, when there are many switchings between arbitrary orders, we have the following structure presented in Fig. 5.3. The reductive-switching order case occurs when the initial chain of derivatives is reduced according to changing the variable-order.

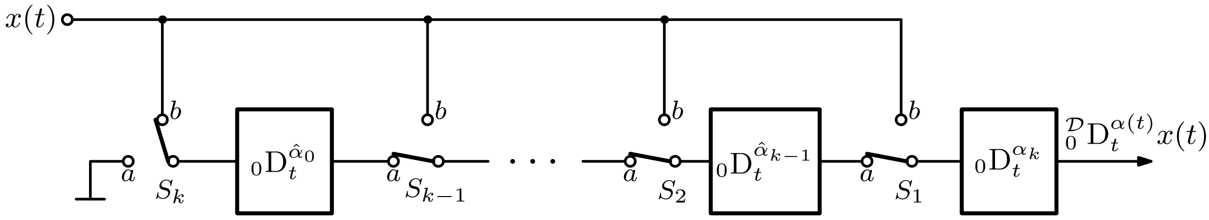


Fig. 5.3. Structure of multiple reductive-switching order derivative ${}^{\text{r-s}}_{0}\mathrm{D}^{\alpha(t)}_{t} f(t)$ (configuration before switch between orders α_0 and α_1)

Using an natural extension of Lemma 5.5 the multiple reductive-switching order case can be achieved.

Lemma 5.6. *[88] For a multiple reductive-switching order case, when the switch between orders $\alpha_0, \ldots, \alpha_k$ occurs every ih time instant, the numerical scheme has the following form:*

$$\begin{pmatrix} {}_0^{\text{r-s}}\mathrm{D}_0^{\alpha(t)} f(0) \\ {}_0^{\text{r-s}}\mathrm{D}_h^{\alpha(t)} f(h) \\ \vdots \\ {}_0^{\text{r-s}}\mathrm{D}_{kh}^{\alpha(t)} f(kh) \end{pmatrix} = \lim_{h\to 0} \prod_{i=0}^{k} W(\hat{\alpha}_{k-i}, k, k-i) \begin{pmatrix} f(0) \\ f(h) \\ \vdots \\ f(kh) \end{pmatrix},$$

where

$$W(\hat{\alpha}_{k-i}, k, k-i) = \begin{pmatrix} W(\hat{\alpha}_{k-i}, k-i) & 0_{k-i+1,i} \\ 0_{i,k-i+1} & I_{i,i} \end{pmatrix},$$

and

$$\alpha(t) = \begin{cases} \alpha_{i+1} + \hat{\alpha}_i & \textit{for} \quad 0 \le t < (i+1)h, \\ \alpha_{i+1} & \textit{for} \quad t \ge (i+1)h, \end{cases} \qquad i = 0, \ldots, k-1.$$

Based on Lemma 5.7 and Lemma 5.8, the equivalence of the $\mathcal{D}$-type operator and the multiple reductive-switching scheme can be proved:

Lemma 5.7. *[88] The product of matrix forms equivalent to the $\mathcal{A}$- and $\mathcal{D}$-type derivatives for negative value of orders gives an identity matrix*

$$^{\mathcal{A}}W(-\alpha, k) \prod_{j=0}^{k} \mathfrak{Q}(\alpha_{k-j}, k-j) = I_{k+1,k+1}. \tag{5.8}$$

Proof. Let us write the matrix form of $\mathcal{A}$-type operator for negative value of orders, then

$$^{\mathcal{A}}W(-\alpha, k) = \begin{pmatrix} h^{\alpha_0} & 0 & 0 & \ldots & 0 & 0 \\ h^{\alpha_1} w_{-\alpha_1,1} & h^{\alpha_1} & 0 & \ldots & 0 & 0 \\ h^{\alpha_2} w_{-\alpha_2,2} & h^{\alpha_2} w_{-\alpha_2\,1} & h^{\alpha_2} & \ldots & 0 & 0 \\ h^{\alpha_3} w_{-\alpha_3,3} & h^{\alpha_3} w_{-\alpha_3,2} & h^{\alpha_3} w_{-\alpha_3,1} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ h^{\alpha_{k-1}} w_{-\alpha_{k-1},k-1} & h^{\alpha_{k-1}} w_{-\alpha_{k-1},k-2} & h^{\alpha_{k-1}} w_{-\alpha_{k-1},k-3} & \ldots & h^{\alpha_{k-1}} & 0 \\ h^{\alpha_k} w_{-\alpha_k,k} & h^{\alpha_k} w_{-\alpha_k,k-1} & h^{\alpha_k} w_{-\alpha_k,k-2} & \ldots & h^{\alpha_k} w_{-\alpha_k,1} & h^{\alpha_k} \end{pmatrix}.$$

The first matrix product can be described as the following block matrices:

$$\begin{aligned} ^{\mathcal{A}}W(-\alpha, k)\mathfrak{Q}(\alpha_k, k) &= \left(\begin{array}{c|c} ^{\mathcal{A}}W(-\alpha, k-1) & 0_{k,1} \\ \hline R(-\alpha, k) & h^{\alpha_k} \end{array}\right) \left(\begin{array}{c|c} I_{k,k} & 0_{k,1} \\ \hline q_{1,k} & h^{-\alpha_k} \end{array}\right) \\ &= \left(\begin{array}{c|c} ^{\mathcal{A}}W(-\alpha, k-1) & 0_{k,1} \\ \hline 0_{1,k} & 1 \end{array}\right), \end{aligned}$$

where

$$R(-\alpha,i) = \begin{pmatrix} h^{\alpha_i} w_{-\alpha_i,i} & h^{\alpha_i} w_{-\alpha_i,i-1} & \cdots & h^{\alpha_i} w_{-\alpha_i,1} \end{pmatrix},$$
$$h^{\alpha_i} q_{1,i} = (-h^{\alpha_i} w_{-\alpha_i,i}, -h^{\alpha_i} w_{-\alpha_i,i-1}, \ldots, -h^{\alpha_i} w_{-\alpha_i,1}).$$

The second matrix product can be expressed by

$$\begin{aligned}
&{}^{\mathcal{A}}W(-\alpha,k)\mathfrak{Q}(\alpha_k,k)\mathfrak{Q}(\alpha_{k-1},k-1)\\
&= \left(\begin{array}{cc|c} {}^{\mathcal{A}}W(-\alpha,k-2) & 0_{k-1,1} & 0_{k-1,1} \\ R(-\alpha,k-1) & h^{\alpha_{k-1}} & 0 \\ \hline 0_{1,k-1} & 0 & 1 \end{array}\right) \left(\begin{array}{cc|c} I_{k-1,k-1} & 0_{k-1,1} & 0_{k-1,1} \\ q_{1,k-1} & h^{-\alpha_{k-1}} & 0 \\ \hline 0_{1,k-1} & 0 & 1 \end{array}\right)\\
&= \left(\begin{array}{cc|c} {}^{\mathcal{A}}W(-\alpha,k-2) & 0_{k-1,1} & 0_{k-1,1} \\ 0_{1,k-1} & 1 & 0 \\ \hline 0_{1,k-1} & 0 & 1 \end{array}\right).
\end{aligned}$$

For general case, when $j=r$ we have the following matrix product

$$\begin{aligned}
&\left({}^{\mathcal{A}}W(-\alpha,k)\prod_{j=0}^{r-1}\mathfrak{Q}(\alpha_{k-j},k-j)\right)\mathfrak{Q}(\alpha_{k-r},k-r)\\
&= \left(\begin{array}{cc|c} {}^{\mathcal{A}}W(-\alpha,k-r-1) & 0_{k-r,1} & 0_{k-r,r} \\ R(-\alpha,k-r) & h^{\alpha_{k-r}} & 0 \\ \hline 0_{r,k-r} & 0_{r,1} & I_{r,r} \end{array}\right) \left(\begin{array}{cc|c} I_{k-r,k-r} & 0_{k-r,1} & 0_{k-r,r} \\ q_{1,k-r} & h^{-\alpha_{k-r}} & 0_{1,r} \\ \hline 0_{r,k-r} & 0_{r,1} & I_{r,r} \end{array}\right)\\
&= \left(\begin{array}{cc|c} {}^{\mathcal{A}}W(-\alpha,k-r-1) & 0_{k-r,1} & 0_{k-r,r} \\ 0_{1,k-r} & 1 & 0 \\ \hline 0_{r,k-r} & 0_{r,1} & I_{r,r} \end{array}\right).
\end{aligned}$$

Finally, for $j=k$ matrix product equals to

$$\begin{aligned}
\left({}^{\mathcal{A}}W(-\alpha,k)\prod_{j=0}^{k-1}\mathfrak{Q}(\alpha_{k-j},k-j)\right)\mathfrak{Q}(\alpha_0,0) &= \left(\begin{array}{c|c} h^{\alpha_0} & 0_{1,k} \\ \hline 0_{k,1} & I_{k,k} \end{array}\right)\left(\begin{array}{c|c} h^{-\alpha_0} & 0_{1,k} \\ \hline 0_{k,1} & I_{k,k} \end{array}\right)\\
&= \left(\begin{array}{c|c} 1 & 0_{1,k} \\ \hline 0_{k,1} & I_{k,k} \end{array}\right),
\end{aligned}$$

consecutive multiplying all of terms ends the proof. ■

Lemma 5.8. *[88] The product of multi-switching numerical scheme equivalent to the $\mathcal{A}$-type derivative and the multiple reductive-switching scheme for negative value of orders gives an identity matrix.*

$$\underbrace{\left(\prod_{j=0}^{k} W(-\hat{\alpha}_j, k, j)\right)}_{{}^{\mathcal{A}}W(-\alpha,k)} \left(\prod_{j=0}^{k} W(\hat{\alpha}_{k-j}, k, k-j)\right) = I_{k+1,k+1}.$$

Proof. By extension of above terms and using connectivity law the following holds

$$\prod_{j=0}^{k-1} W(-\hat{\alpha}_j, k, j) \underbrace{W(-\hat{\alpha}_k, k, k) W(\hat{\alpha}_k, k, k)}_{I_{k+1,k+1}} \prod_{j=1}^{k} W(\hat{\alpha}_{k-j}, k, k-j) =$$

$$\prod_{j=0}^{k-2} W(-\hat{\alpha}_j, k, j) \underbrace{W(-\hat{\alpha}_{k-1}, k, k-1) W(\hat{\alpha}_{k-1}, k, k-1)}_{I_{k+1,k+1}} \prod_{j=2}^{k} W(\bar{\alpha}_{k-j}, k, k-j) =$$

$$\prod_{j=0}^{k-l+1} W(-\hat{\alpha}_j, k, j) \underbrace{W(-\hat{\alpha}_{k-l}, k, k-l) W(\hat{\alpha}_{k-l}, k, k-l)}_{I_{k+1,k+1}} \prod_{j=l-1}^{k} W(\hat{\alpha}_{k-j}, k, k-j) =$$

$$\cdots = I_{k+1,k+1}.$$

■

Finally, the following theorem can be formulated.

Theorem 5.9. *[88] The multiple reductive-switching order scheme presented in Fig. 5.3 is equivalent to the $\mathcal{D}$-type variable-order derivative given by Def. 5.1.*

Proof. The proof is a conclusion directly based on Lemma 5.7 and Lemma 5.8.

$${}^{\mathcal{A}}W(-\alpha, k) \prod_{j=0}^{k} \mathfrak{Q}(\alpha_{k-j}, k-j) = {}^{\mathcal{A}} W(-\alpha, k) \prod_{j=0}^{k} W(\hat{\alpha}_{k-j}, k, k-j),$$

thus,

$$\prod_{j=0}^{k} \mathfrak{Q}(\alpha_{k-j}, k-j) = \prod_{j=0}^{k} W(\hat{\alpha}_{k-j}, k, k-j),$$

where $\alpha_j = \sum_{i=j}^{k} \hat{\alpha}_i$. ■

Lemma 5.7 and Lemma 5.8 reveal an interesting property of the fractional variable-order operators, so-called the duality property. It specifies that the composition of two particular types of fractional variable-order operator with negative value of orders gives an original function.

Remark 5.10. [78, 88] The duality property between $\mathcal{A}$ and $\mathcal{D}$-types operators.

$$ {}_0^{\mathcal{A}}\mathrm{D}_t^{\alpha(t)}\left({}_0^{\mathcal{D}}\mathrm{D}_t^{-\alpha(t)}f(t)\right) = {}_0^{\mathcal{D}}\mathrm{D}_t^{\alpha(t)}\left({}_0^{\mathcal{A}}\mathrm{D}_t^{-\alpha(t)}f(t)\right) = f(t). $$

Some additional properties and modifications of the $\mathcal{D}$-type operator appear in [72], where for e.g. it is successfully used to eliminate the steady-state error in control system.

5.4 Initial conditions for the $\mathcal{D}$-type operator

Considering the initial conditions in the $\mathcal{D}$-type fractional variable-order definition is motivated by similar reasons as in Section 2.4. The indispensable initial voltage occurring in fractional order impedances has an impact on experimental results.

This section contains the numerical scheme for considering the initial conditions into the $\mathcal{D}$-type fractional variable-order definition in the form of finite and infinite length. Let us remind the $\mathcal{D}$-type recursive difference for time k^* [71, 91]:

$$ {}_0^{\mathcal{D}}\Delta_{k^*}^{\alpha_{k^*}} x_{k^*} = \frac{x_{k^*}}{h^{\alpha_{k^*}}} - \sum_{j=1}^{k^*}(-1)^j \binom{-\alpha_{k^*}}{j} {}_0^{\mathcal{D}}\Delta_{k^*-j}^{\alpha_{k^*}} x_{k^*-j} \,. $$

Again, we assume that the zero time is shifted to some point $t^* = T$. Which can be interpreted that the system has some initial energy (history). In this case, the difference can be rewritten in the following form for $k = k^* - T$ [71, 91]:

$$ {}_{-T}^{\mathcal{D}}\Delta_k^{\alpha_k} x_k = \frac{x_k}{h^{\alpha_k}} - \sum_{j=1}^{k+T}(-1)^j \binom{-\alpha_k}{j} {}_{-T}^{\mathcal{D}}\Delta_{k-j}^{\alpha_k} x_{k-j} \,. \tag{5.9} $$

Then, (5.9) can be rewritten to the form [71, 91]

$$ \begin{aligned} {}_{-T}^{\mathcal{D}}\Delta_k^{\alpha_k} x_k = \frac{x_k}{h^{\alpha_k}} &- \sum_{j=1}^{k-1}(-1)^j \binom{-\alpha_k}{j} {}_{-T}^{\mathcal{D}}\Delta_{k-j}^{\alpha_k} x_{k-j} \\ &- \sum_{j=k}^{k+T}(-1)^j \binom{-\alpha_k}{j} {}_{-T}^{\mathcal{D}}\Delta_{k-j}^{\alpha_k} x_{k-j} \,. \end{aligned} \tag{5.10} $$

It is noticeable, that from practical point of view, the fractional order differences from ${}^{\mathcal{D}}\Delta^{\alpha_0} x_0$ to ${}^{\mathcal{D}}\Delta^{\alpha_0} x_{-T}$ can be assumed as initial conditions statement for $\mathcal{D}$-type fractional order difference. In this case, the initial conditions statement is a series from ${}^{\mathcal{D}}\Delta^{\alpha_k} x_0$ to ${}^{\mathcal{D}}\Delta^{\alpha_k} x_{-T}$.

Constant, finite-time initial conditions case

When the system is in a steady state before, for example, control algorithm starts ($t < 0$), we can admit that ${}^{\mathcal{D}}_{-T}\Delta^{\alpha_i}_{i} x_i = c = \text{const}$ for $i = -T \dots 0$. In this case, we can obtain [71, 91]

$$ {}^{\mathcal{D}}_{-T}\Delta^{\alpha_k}_{k} x_k = \frac{x_k}{h^{\alpha_k}} - \sum_{j=1}^{k-1} (-1)^j \binom{-\alpha_k}{j} {}^{\mathcal{D}}_{-T}\Delta^{\alpha_{k-j}}_{k-j} x_{k-j} - \sum_{j=k}^{k+T} (-1)^j \binom{-\alpha_k}{j} c. \tag{5.11} $$

Constant, infinite-time initial conditions case

For infinite length of initial conditions, (5.11) can be rewritten to the form of the following theorem:

Theorem 5.11. *[71, 91] The $\mathcal{D}$-type of fractional order difference for initial conditions in the form of infinite length constant function ${}^{\mathcal{D}}\Delta^{\alpha_i} x_i = c = const$ for $i = -\infty \dots 0$ is given by the following relation:*

$$ {}^{\mathcal{D}}_{-\infty}\Delta^{\alpha_k}_{k} x_k = \frac{x_k}{h^{\alpha_k}} - \sum_{j=1}^{k-1} (-1)^j \left[\binom{-\alpha_k}{j} {}^{\mathcal{D}}_{-\infty}\Delta^{\alpha_k}_{k-j} x_{k-j} - c \right] + c. \tag{5.12} $$

Proof. The second sum in (5.10), for infinite length of initial conditions $T \to \infty$ (what implies that also $T + k \to \infty$), can be obtained in the following form

$$ \sum_{j=0}^{\infty} (-1)^j \binom{-\alpha_k}{j} c = \sum_{j=0}^{k-1} (-1)^j \binom{-\alpha_k}{j} c + \sum_{j=k}^{k+T} (-1)^j \binom{-\alpha_k}{j} c \,. $$

Using Remark 2.10 we obtain

$$ \begin{aligned} \sum_{j=k}^{k+T} (-1)^j \binom{-\alpha_k}{j} c &= \sum_{j=0}^{\infty} (-1)^j \binom{-\alpha_k}{j} c - \sum_{j=0}^{k-1} (-1)^j \binom{-\alpha_k}{j} c \\ &= -c - \sum_{j=1}^{k-1} (-1)^j \binom{-\alpha_k}{j} c \,, \end{aligned} $$

and combining with (5.11) finishes the proof. ■

Continuous-time case

In the continuous-time domain the scheme for including the initial conditions in the form of infinite length can be introduced as:

Theorem 5.12. *[71, 91] The $\mathcal{D}$-type fractional variable-order derivative for initial conditions in the form of infinite length constant function ${}^{\mathcal{D}}_{-\infty}\mathrm{D}^{\alpha(t)}_{t} f(t) = c = const$ for*

$t = (-\infty, 0)$ *is given by the following relation:*

$$ {}^{\mathcal{D}}_{-\infty}\mathrm{D}^{\alpha(t)}_{t} f(t) = \frac{f(t)}{h^{\alpha(t)}} - \sum_{j=1}^{k-1} (-1)^j \left[\binom{-\alpha(t)}{j} {}^{\mathcal{D}}_{-\infty}\mathrm{D}^{\alpha(t)}_{t-jh} f(t) - c \right] + c. $$

Proof. The proof can be done in similar way as proof of Theorem 5.11. ■

5.5 Experimental setup of simple switching order case

Analog model of switching system, used in experimental setup, corresponding to the switching scheme given in Fig. 5.2, is presented in Fig 5.4. Due to the use of operational amplifiers, the signals with inverted polarization are obtained. So, the analog realization require the additional amplifiers with gain -1 for each integrator (amplifiers A_2 and A_4). In general cases, the scheme based on amplifiers A_1 and A_3 contains resistors R and impedances Z_1, Z_2 chosen according to the type of realized orders. As a realization of switches S_1 and S_2, integrated analog switches DG303 are used. In order to obtain impedance order, which equal to 0.5 or 0.25, the domino-ladder approximations are used. The experimental circuit is connected to the dSPACE DS1104 PPC card with a PC. At the beginning both integrators are connected in series (switch S_1 is in position a, and switch S_2 is in position b), and after order switching, in time T, the integrator at beginning of the chain (based on impedance Z_1) is disconnected. So, switch S_1 is in position b, and switch S_2 is in position a.

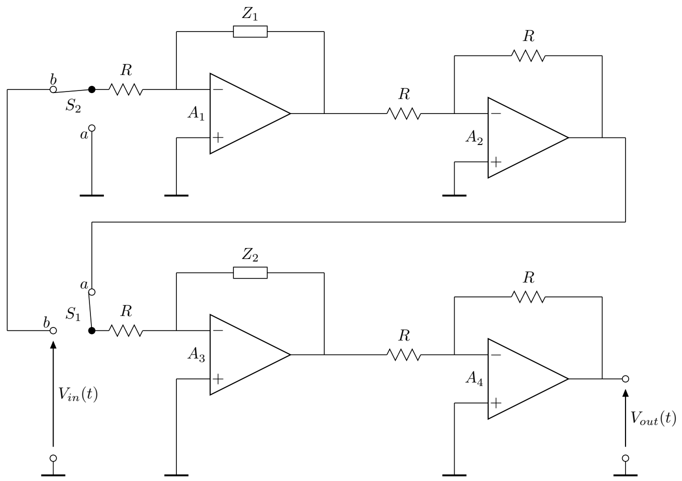


Fig. 5.4. Analog realization of the $\mathcal{D}$-type of fractional variable-order integrator.

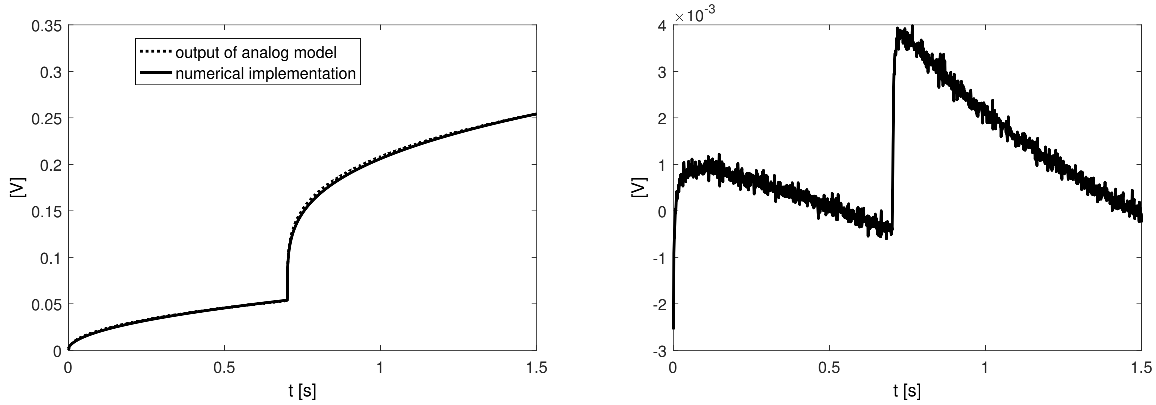


Fig. 5.5. Results of analog and numerical implementation of the $\mathcal{D}$-type fractional variable-order integrator for switching order from $\alpha = 0.5$ to $\alpha = 0.25$ (*left*) and their modeling error (*right*).

5.5.1 Results of the $\mathcal{D}$-type integrators

Example 5.13. [88] Integrator with order switching from $\alpha = 0.5$ to $\alpha = 0.25$.

In this case, the scheme shown in Fig. 5.4, contains the following structure: Z_1 and Z_2 are the quarter-order integrators, $R = 100\text{k}\Omega$. The identification results are obtained by numerical minimization of time responses square error with sampling time 0.001 s and input signal being $u(t) = 1(t)$ V.

After identification process, the following models for orders -0.5 and -0.25 are obtained, respectively:

$$y(t) = {}_0\mathrm{D}_t^{-0.5} a_1 u(t) = 0.0567 {}_0\mathrm{D}_t^{-0.5} u(t),$$
$$y(t) = {}_0\mathrm{D}_t^{-0.25} a_2 u(t) = 0.2359 {}_0\mathrm{D}_t^{-0.25} u(t),$$

which gives rise to the following variable-order integrator:

$$y(t) = {}_0^{\mathcal{D}}\mathrm{D}_t^{-\alpha(t)} \left[a(t) u(t) \right],$$

where (for the switching time $T = 0.7$ s)

$$a(t) = \begin{cases} 5.6719 & \text{for } t < 0.7, \\ 23.5856 & \text{for } t \geq 0.7, \end{cases}$$

and

$$\alpha(t) = \begin{cases} 0.5 & \text{for } t < 0.7, \\ 0.25 & \text{for } t \geq 0.7. \end{cases}$$

The experimental results compared to numerical implementation of the $\mathcal{D}$-type variable-order integrator are presented in Fig. 5.5.

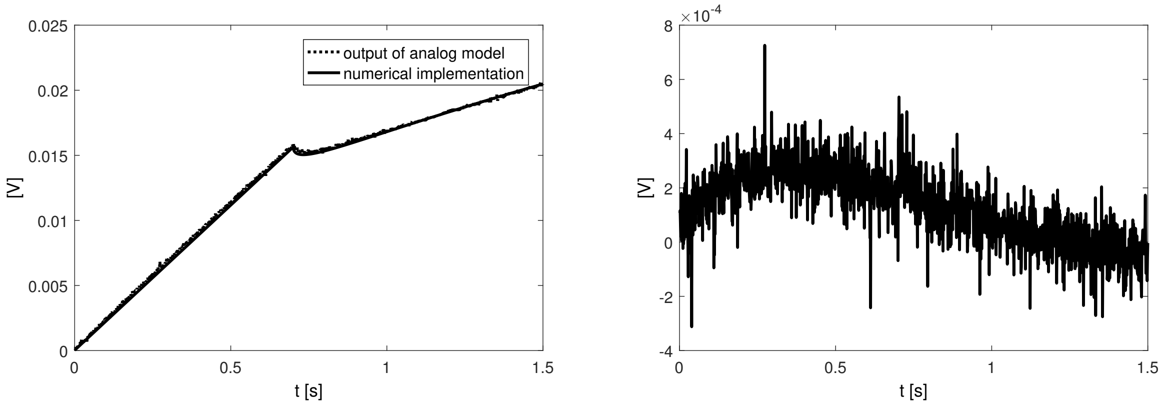


Fig. 5.6. Results of analog and numerical implementation of the $\mathcal{D}$-type fractional variable-order integrator for switching order from $\alpha = 1$ to $\alpha = 0.5$ (*left*) and their modeling error (*right*).

Example 5.14. [88] Integrator with order switching from $\alpha = 1$ to $\alpha = 0.5$.

In this configuration, the scheme shown in Fig. 5.4, contains the following structure: Z_1 and Z_2 are the half order integrators, $R = 100\text{k}\Omega$. The identification results are obtained by numerical minimization of time responses square error with sampling time $T_\text{s} = 0.001$ s and input signal being $u(t) = 0.01 \cdot 1(t)$ V.

After identification process, the following models for order -1 and -0.5 are obtained, respectively:

$$
\begin{aligned}
y(t) &= {}_0\text{D}_t^{-1} a_1 u(t) = 2.23 {}_0\text{D}_t^{-1} u(t), \\
y(t) &= {}_0\text{D}_t^{-0.5} a_2 u(t) = 1.4940 {}_0\text{D}_t^{-0.5} u(t),
\end{aligned}
$$

which gives rise to the following variable-order integrator:

$$
y(t) = {}_0^{\mathcal{D}}\text{D}_t^{-\alpha(t)} \left[a(t) u(t) \right],
$$

where (for the switching time $T = 0.7$ s)

$$
a(t) = \begin{cases} 2.23 & \text{for } t < 0.7, \\ 1.4940 & \text{for } t \geq 0.7, \end{cases}
$$

and

$$
\alpha(t) = \begin{cases} 1 & \text{for } t < 0.7, \\ 0.5 & \text{for } t \geq 0.7. \end{cases}
$$

The experimental results compared to numerical implementation of the $\mathcal{D}$-type variable-order integrator are presented in Fig. 5.6.

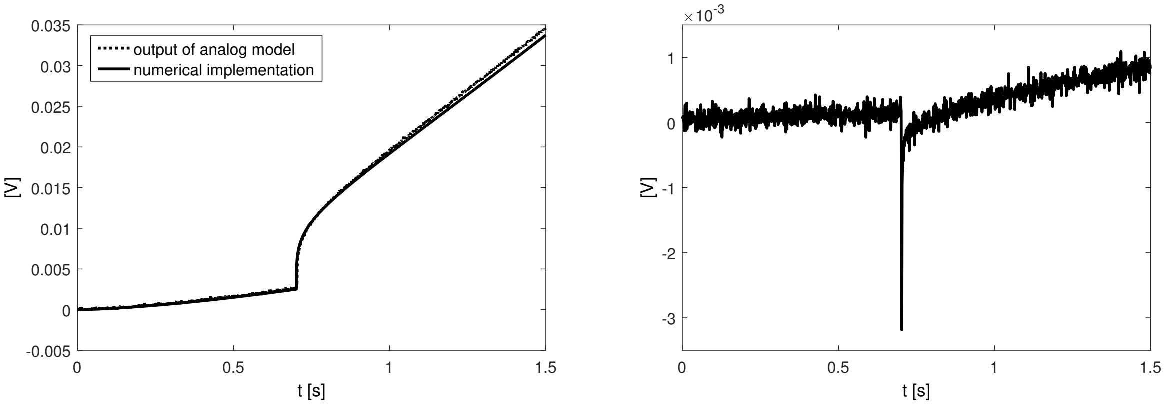


Fig. 5.7. Results of analog and numerical implementation of the $\mathcal{D}$-type fractional variable-order integrator for switching order from $\alpha = 0.5$ to $\alpha = 0.25$ (*left*) and their modeling error (*right*).

Example 5.15. [88] Integrator with order switching from $\alpha = 0.5$ to $\alpha = 0.25$, time response on a ramp function.

In this example, configuration of the experimental setup is similar to the one already used in Example 5.13. The identification results are obtained by numerical minimization of time responses square error with sampling time $T_s = 0.001$ s and input signal being a ramp function $u(t) = 0.1 \cdot t \cdot 1(t)$ V.

After identification process, the following models for orders -0.5 and -0.25 are obtained, respectively:

$$y(t) = {}_0\mathrm{D}_t^{-0.5} a_1 u(t) = 0.057 {}_0\mathrm{D}_t^{-0.5} u(t),$$
$$y(t) = {}_0\mathrm{D}_t^{-0.25} a_2 u(t) = 0.2358 {}_0\mathrm{D}_t^{-0.25} u(t),$$

which gives rise to the following variable-order integrator:

$$y(t) = {}_0^{\mathcal{D}}\mathrm{D}_t^{-\alpha(t)} \left[a(t)u(t)\right],$$

where (for the switching time $T = 0.7$ s)

$$a(t) = \begin{cases} 0.057 & \text{for } t < 0.7, \\ 0.2358 & \text{for } t \geq 0.7, \end{cases}$$

and

$$\alpha(t) = \begin{cases} 0.5 & \text{for } t < 0.7, \\ 0.25 & \text{for } t \geq 0.7. \end{cases}$$

The experimental results compared to numerical implementation of the $\mathcal{D}$-type variable-order integrator are presented in Fig. 5.7.

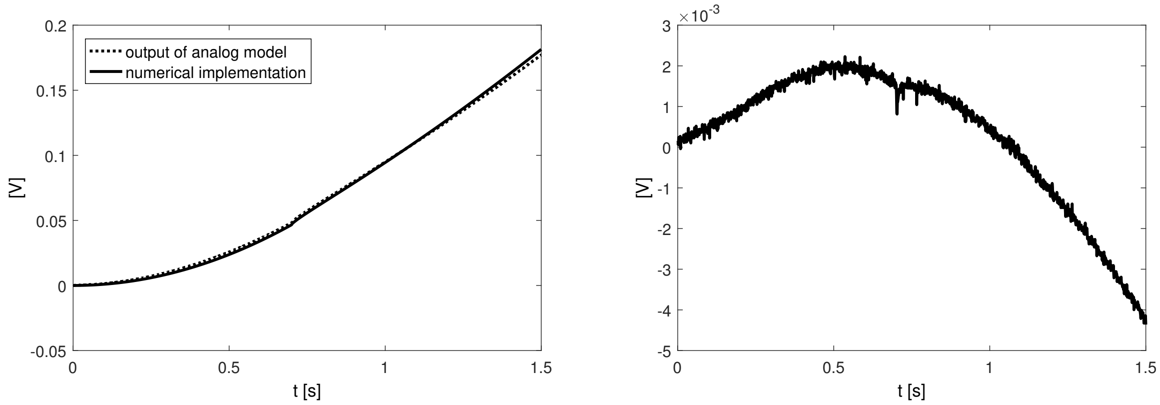


Fig. 5.8. Results of analog and numerical implementation of the $\mathcal{D}$-type fractional variable-order integrator for switching order from $\alpha = 1$ to $\alpha = 0.5$ (*left*) and their modeling error (*right*).

Example 5.16. [88] Integrator with order switching from $\alpha = 1$ to $\alpha = 0.5$, time response on a ramp function.

In this example, configuration of the experimental setup is similar to the one already used in Example 5.14. The identification results are obtained by numerical minimization of time responses square error with sampling time $T_s = 0.001$ s and input signal being ramp function $u(t) = 0.1 \cdot t \cdot 1(t)$ V.

After identification process, the following models for orders -1 and -0.5 are obtained, respectively:

$$y(t) = {}_0\mathrm{D}_t^{-1} a_1 u(t) = 1.7824 {}_0\mathrm{D}_t^{-1} u(t),$$
$$y(t) = {}_0\mathrm{D}_t^{-0.5} a_2 u(t) = 1.3514 {}_0\mathrm{D}_t^{-0.5} u(t).$$

which gives rise to the following variable-order integrator:

$$y(t) = {}_0^{\mathcal{D}}\mathrm{D}_t^{-\alpha(t)} \left[a(t)u(t)\right],$$

where (for the switching time $T = 0.7$ s)

$$a(t) = \begin{cases} 0.17824 & \text{for } t < 0.7, \\ 1.3514 & \text{for } t \geq 0.7, \end{cases}$$

and

$$\alpha(t) = \begin{cases} 1 & \text{for } t < 0.7, \\ 0.5 & \text{for } t \geq 0.7. \end{cases}$$

The experimental results compared to numerical implementation of the $\mathcal{D}$-type variable-order integrator are presented in Fig. 5.8.

5.5.2 Results of the $\mathcal{D}$-type inertial systems with initial conditions

Realization of the fractional variable-order inertial system based on fractional variable-order integral system presented in Fig. 5.4 is shown in Fig. 5.9. The order of the system depends on Z_1, Z_2 impedances and position of S_1 and S_2 switches presented in Fig. 5.4. When S_1 and S_2 switches do not change their positions during experiment, then system is considering as a fractional constant-order inertial system.

To gather data for fractional variable-order inertial system with initial conditions the domino-ladder structure is firstly loaded to the initial value of voltage, and then the scheme presented in 5.9 is running with $u(t) = 0$.

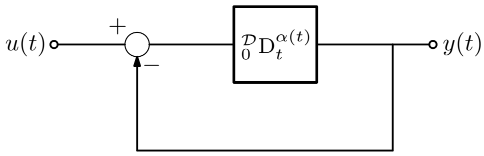


Fig. 5.9. Realization of the $\mathcal{D}$-type fractional variable-order inertial system based on fractional variable-order integral system presented in Fig. 5.4.

Example 5.17. [71, 91] Inertial system with order switching from -0.5 to -0.25.

In this case, the scheme presented in Fig. 5.4 takes the following structure: Z_1, Z_2 are quarter-order impedances, resistors $R = 5\text{k}\Omega$ and until switching time the S_1 switch is connected to terminal marked as a and S_2 switch is connected to terminal b. After switching time ($T \geq 0.7$ s) switches change their positions. The initial value of voltage equals to 0.2 V and the sample time is chosen to 0.001 s.

The identification parameters for half- and quarter-order integrators are shown in Fig 2.3 and in Fig. 2.4. So, identified variable-order system has the following form:

$$y(t) = {}^{\mathcal{D}}_{-\infty}\mathrm{D}^{\alpha(t)}_{t}\left[a(t)(u(t) - y(t))\right],$$

where (for the switching time $T = 0.7$ s)

$$a(t) = \begin{cases} 20 & \text{for } t < 0.7, \\ 1.55 & \text{for } t \geq 0.7, \end{cases}$$

and

$$\alpha(t) = \begin{cases} -0.5 & \text{for } t < 0.7, \\ -0.25 & \text{for } t \geq 0.7. \end{cases}$$

The experimental results of fractional variable-order inertial system with order switching between -0.5 to -0.25 at time $T = 0.7$ s, compared to the numerical results, are presented in Fig. 5.10.

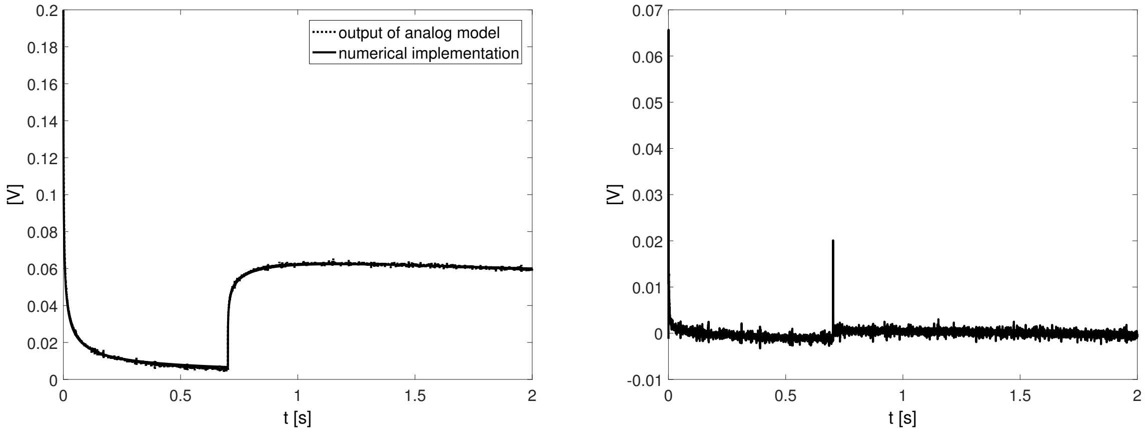


Fig. 5.10. Results of analog and numerical implementation of the $\mathcal{D}$-type inertial system with initial condition equals to 0.2 V (*left*) and their modeling error (*right*).

5.6 Experimental setup of multiple-switching analog model

The analog model of $\mathcal{D}$-type derivative implemented directly as the switching strategy given in Fig. 5.3 gives a possibility to switch order only once. However, for both impedances Z_1 and Z_2 being the half-order impedances this integrator can be extended to multi-switching structure, and then allows to change the order between $\alpha = 1$ and $\alpha = 0.5$ in any time. As it can be noticed, before order switching, voltage on impedance Z_2 is the same as for integrator of order $\alpha = 1$. This means that the half-order impedance is charged by the same voltage as a first order integrator. After switching order to $\alpha = 0.5$, the first integrator, based on impedance Z_1, is took out and the signal comes directly to the second integrator (based on Z_2). This means, that before switching time we can charge only one half order impedance with the voltage originating from the first order integrator. After switching time we can use integrator, charged by the voltage of domino-ladder. Based on it, the multiple switching analog model can be build.

5.6.1 Realization of the multiple-switching analog model

Multiple-switching analog model, designed according to $\mathcal{D}$-type fractional variable-order integral is presented in Fig. 5.11. The experimental setup contains:

- dSPACE DS1104 PPC card with a PC,
- operational amplifiers TL071,
- analog switches DG303,
- passive elements such as:
  - resistors with the following values: $R = 750\text{k}\Omega$, $R_1 = 2.4\text{k}\Omega$, $R_2 = 8.2\text{k}\Omega$,

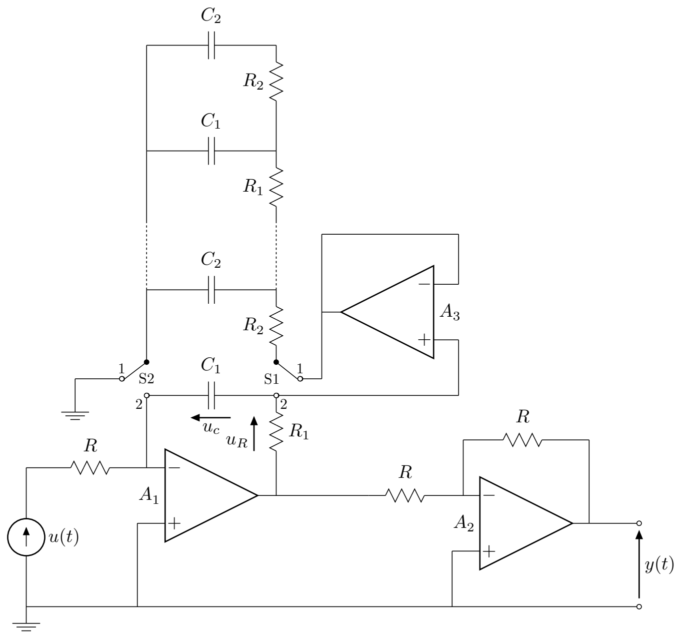


Fig. 5.11. Multiple-switching analog realization of the $\mathcal{D}$-type fractional variable-order integrator.

– capacitors with the following values: $C_1 = 330\text{nF}$, $C_2 = 220\text{nF}$.

Depending on switches position marked as S in Fig. 5.11, the circuit can be described by fractional order integral system ($\alpha = 0.5$) or traditional integral system ($\alpha = 1$).

1. For a case, when S-switches are connected to terminals marked as 1, the following fractional order derivative function has been obtained:

$$y_1(t) = a_{10}\mathrm{D}_t^{-1}u(t), \tag{5.13}$$

   where a_1 is a time constant.

2. For a case, when S-switches are connected to terminals marked as 2, the following fractional order derivative function has been obtained:

$$y_2(t) = a_{20}\mathrm{D}_t^{-0.5}u(t), \tag{5.14}$$

where a_2 is a time constant of the half order integral; it should be stressed that $T_2 \neq T_1$.

To invert polarization of output signal, the A_3 operational amplifier are used.

Based on switches position, the system can be switched in two ways:

1. switching from terminals 1 to 2: In this case, the system described by the first order integral $y_1(t)$ is switched to system of order half described by function $y_2(t)$. To keep the behavior of the $\mathcal{D}$-type definition, it is necessary to maintain a continuous voltage of capacitors in the rest branches, between terminals marked as 1. The voltages of

capacitors are set to the value of voltage on the capacitor in the first order integral system (the first capacitor). The main idea has been shown in Fig. 5.11. The initial value of the half order analog model are values of voltage on capacitors in all branches. In the proposed realization of the switching order method, capacitors in branches that are not used in the first order process are charged to the value obtained on the first branch (first order integrator) by voltage follower A_3;

2. switching from terminals 2 to 1:

   In this case the system described by function $y_2(t)$ (half order) is switched to system described by the function $y_1(t)$ (first order). In this configuration, branches in closed-loop are connected to terminals marked as 2, and after switching, the terminals are changed to 1.

It is worth to notice that the presented realization is the multiple-switching analog model. Position of switches (S_1 and S_2) can be change in any time t.

Example 5.18. [89] Results of the multiple-switching integrator.

The identification results for multiple-switching integrator are obtained by numerical minimization of time responses square error with sampling time 0.001 s and input signal being $u(t) = 0.1 \cdot 1(t)$ V.

After identification process, the following models for orders -0.5 and -1 are obtained:

$$y(t) = {}_0\mathrm{D}_t^{-0.5} a_1 u(t) = 0.185 {}_0\mathrm{D}_t^{-0.5} u(t),$$
$$y(t) = {}_0\mathrm{D}_t^{-1} a_2 u(t) = 3.7 {}_0\mathrm{D}_t^{-1} u(t),$$

which gives rise to the following variable-order integrator:

$$y(t) = {}_0^{\mathcal{D}}\mathrm{D}_t^{-\alpha(t)} \left[a(t) u(t) \right].$$

The order $\alpha(t)$ and parameter $a(t)$ change their values according to

$$\alpha(t) = \begin{cases} \alpha_1 & \text{for } t \in [0, \frac{1}{2}) \cup [1, \frac{3}{2}), \\ \alpha_2 & \text{for } t \in [\frac{1}{2}, 1) \cup [\frac{3}{2}, 2), \end{cases} \tag{5.15}$$

and

$$a(t) = \begin{cases} a_1 & \text{for } t \in [0, \frac{1}{2}) \cup [1, \frac{3}{2}), \\ a_2 & \text{for } t \in [\frac{1}{2}, 1) \cup [\frac{3}{2}, 2). \end{cases} \tag{5.16}$$

Thus, the switches of the variable-order (and gain coefficients) occur at time-instants $t_i = 0.5i$, $i = 0, \ldots, N$, $N = 4$.

Identification results and the difference between step responses of analog model and its numerical implementation are presented in Fig. 5.12 for $\alpha = 0.5$ order integrator, and

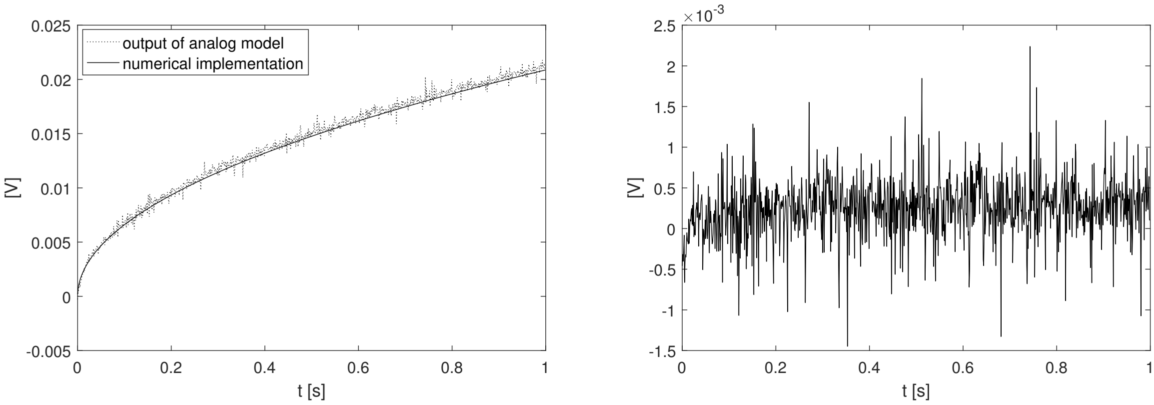


Fig. 5.12. Identification results of constant-order $\alpha = 0.5$ integrator (*left*) and their modeling error (*right*).

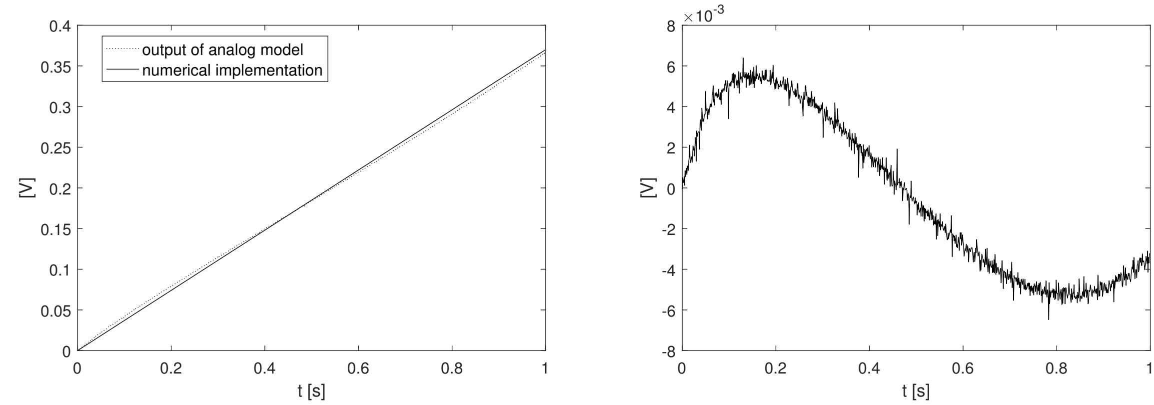


Fig. 5.13. Identification results of constant-order $\alpha = 1$ integrator (*left*) and their modeling error (*right*).

in Fig. 5.13 for $\alpha = 1$ order integrator.

The experimental results compared to numerical implementation of the multiple-switching variable-order integrator based on $\mathcal{D}$-type derivative are presented in Fig. 5.14 and in Fig. 5.15.

5.6.2 Realization of the multiple-switching inertial system

When multiple-switching analog model of fractional variable-order integral presented in Fig. 5.11 appears in forward path in Fig. 5.9, then the multi-switching inertial system equivalent to $\mathcal{D}$-type inertial system can be designed. The order of the system depends on position of S_1 and S_2 switches presented in Fig. 5.11. When S_1 and S_2 switches do not change their positions during experiment, the system is considering as a fractional constant-order inertial system.

Example 5.19. [89] The multiple-switching inertial systems.

The order $\alpha(t)$ and parameter $a(t)$ change their values according to (5.15) and (5.16). So, the parameters' identification of half and first order integrator are presented in Fig. 5.12 and in Fig. 5.13.

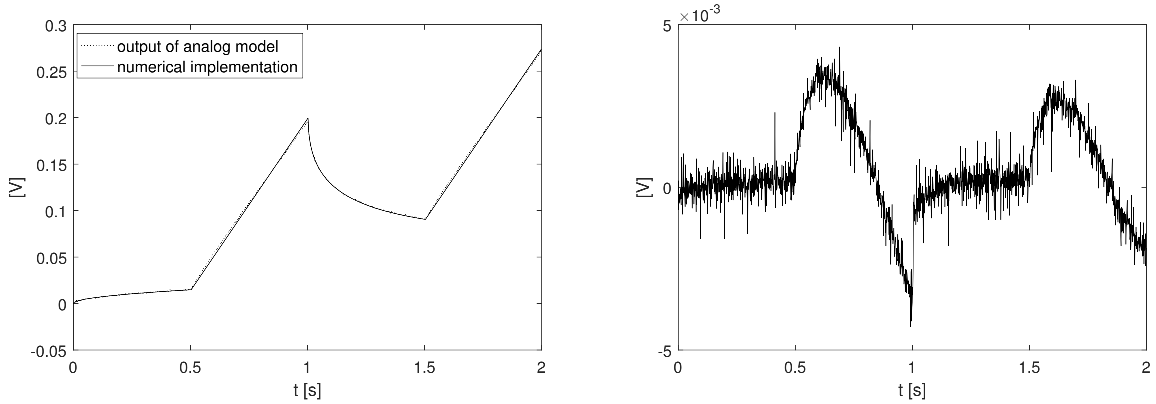


Fig. 5.14. Results of multiple-switching integral analog model and its numerical implementation for $\alpha_1 = 0.5$, $\alpha_2 = 1$, $a_1 = 0.185$, and $a_2 = 3.7$ (*left*); modeling error (*right*).

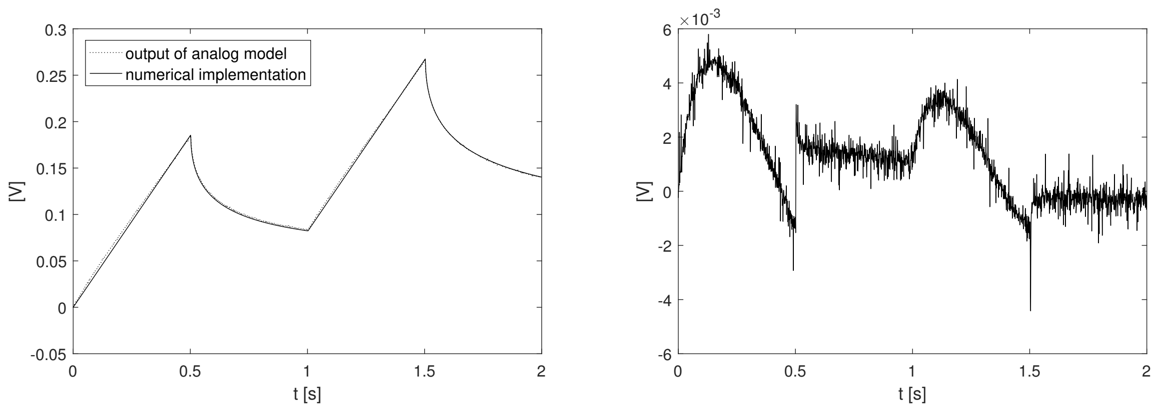


Fig. 5.15. Results of multiple-switching integral analog model and its numerical implementation for $\alpha_1 = 1$, $\alpha_2 = 0.5$, $a_1 = 3.7$ and $a_2 = 0.185$ (*left*); modeling error (*right*).

Thus, the identification results for multiple-switching variable-order inertial system has the following form

$$y(t) = {}^{\mathcal{D}}_{0}\mathrm{D}^{\alpha(t)}_{t} \left[a(t)(u(t) - y(t)) \right].$$

The experimental results compared to numerical implementation of the multiple-switching variable-order inertial system based on $\mathcal{D}$-type integrator are presented in Fig. 5.14 and in Fig. 5.15.

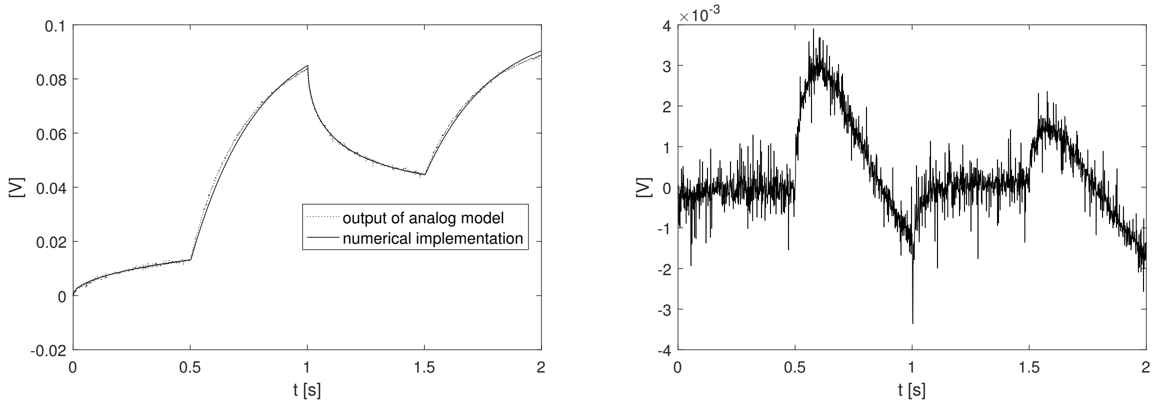


Fig. 5.16. Results of multiple-switching inertial system and its numerical implementation for $\alpha_1 = -0.5$, $\alpha_2 = -1$, $a_1 = 0.185$, and $a_2 = 3.7$ (*left*) and their modeling error (*right*).

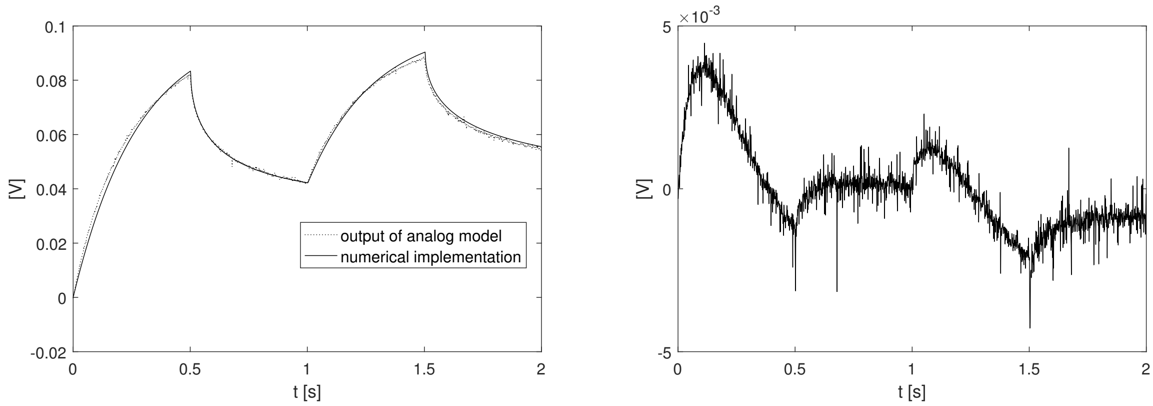


Fig. 5.17. Results of multiple-switching inertial system and its numerical implementation for $\alpha_1 = -1$, $\alpha_2 = -0.5$, $a_1 = 3.7$ and $a_2 = 0.185$ (*left*) and their modeling error (*right*).

5.7 Summary

At the beginning of this chapter the $\mathcal{D}$-type fractional variable-order difference and derivative were introduced. It was proved that the $\mathcal{D}$-type definition can be presented as the reductive-switching order scheme. Then, based on the analogy for such switching scheme its equivalence to the multiple-switching analog model was given. A huge advantage of multiple-switching scheme against to reductive one is a possibility to change the order of $\mathcal{D}$-type derivative between -0.5 and -1 in any time based on invariable structure.

There is no necessary to extend the electrical circuit to the additional operational amplifiers. The method for including initial conditions into $\mathcal{D}$-type definition in the form of finite and infinite time was investigated as well. The second part of this chapter was devoted to validation of theoretical considerations. The experimentally obtained data for reductive-switching and multiple-switching integrals were compared to their numerical implementation. Moreover, the multiple-switching analog model was used to gather the data for $\mathcal{D}$-type inertial systems. At the end, the reductive-switching analog model were used to collect the data for $\mathcal{D}$-type inertial system with initial conditions. All presented experimental results show high accuracy of proposed method.

Chapter 6

The $\mathcal{E}$-type fractional variable-order definition

In this chapter the $\mathcal{E}$-type fractional variable-order difference and derivative are presented together with their matrix forms. Equivalence of numerical switching scheme to the $\mathcal{E}$-type derivative is proved. Moreover, a method for considering the initial conditions into $\mathcal{E}$-type definition is introduced. The order composition properties of $\mathcal{E}$-type derivative with constant-order derivative are proved as well. At the end, to validate the theoretical considerations, the analog models being the realizations of fractional variable-order integral, inertial and inertial with initial conditions systems are build and used to gather the experimental data. Then, the experimental data are compared to their numerical implementations.

6.1 Introduction to $\mathcal{E}$-type fractional variable-order operator

The $\mathcal{E}$-type fractional variable-order operator is related to recursive fractional constant-order definition and can be achieved in the case, when binomial coefficients and corresponding orders are defined in the same manner as in $\mathcal{B}$-type fractional variable-order operator. In this case, the definition has the following form:

Definition 6.1. [31] The $\mathcal{E}$-type fractional variable-order derivative is as follows

$$ {}^{\mathcal{E}}_{0}\mathrm{D}^{\alpha(t)}_{t} f(t) = \lim_{h\to 0}\left(\frac{f(t)}{h^{\alpha(t)}} - \sum_{j=1}^{n}(-1)^j\binom{-\alpha(t-jh)}{j}\frac{h^{\alpha(t-jh)}}{h^{\alpha(t)}}\,{}^{\mathcal{E}}_{0}\mathrm{D}^{\alpha(t)}_{t-jh}f(t)\right), $$

where $n = \lfloor t/h \rfloor$ and h is a sampling time.

Definition 6.2. [31] The discrete realization of $\mathcal{E}$-type fractional variable-order derivative is as follows

$$ {}^{\mathcal{E}}_{0}\Delta^{\alpha_l}_{l} f_l = \frac{f_l}{h^{\alpha_l}} - \sum_{j=1}^{k}(-1)^j\binom{-\alpha_{l-j}}{j}\frac{h^{\alpha_{l-j}}}{h^{\alpha_l}}\,{}_{0}\Delta^{\alpha_{l-j}}_{l-j} f_{l-j}, $$

for $l = 0, 1, 2, \dots, k$.

In Fig. 6.1, plots of step function derivatives (according to Def. 6.1) are presented for $\alpha_1 = -1$, $\alpha_2 = -2$, and

$$\alpha_3(t) = \begin{cases} -1 & \text{for } 0 \le t < 1, \\ -2 & \text{for } 1 \le t \le 2. \end{cases} \tag{6.1}$$

Until switching time the output of $\mathcal{E}$-type derivative (see Fig. 6.1) behaves like a constant-

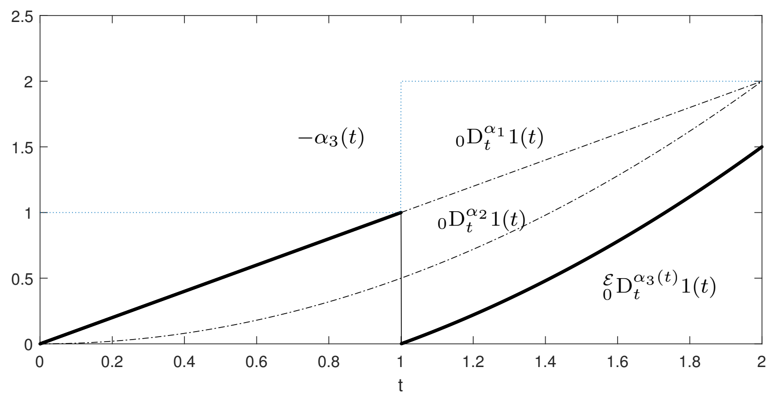


Fig. 6.1. Plots of step function derivatives with respect to the $\mathcal{E}$-type derivative (given by Def. 6.1).

order derivative. The big difference occurs at the switching time instant, then $\mathcal{E}$-type definition keep to integrate with new order beginning with zero initial value.

6.2 Matrix approach for the $\mathcal{E}$-type operator

By extension of Def. 6.2 there is possibility to achieve its matrix representation.

Lemma 6.3. *[31] The $\mathcal{E}$-type fractional difference given by Def. 6.2 can be expressed in the following matrix form:*

$$\begin{pmatrix} {}_0^{\mathcal{E}}\Delta_0^{\alpha_0} f_0 \\ {}_0^{\mathcal{E}}\Delta_1^{\alpha_1} f_1 \\ {}_0^{\mathcal{E}}\Delta_2^{\alpha_2} f_2 \\ \vdots \\ {}_0^{\mathcal{E}}\Delta_k^{\alpha_k} f_k \end{pmatrix} = \prod_{j=0}^{k} \mathfrak{Q}(\alpha_{k-j}, k-j) \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_k \end{pmatrix},$$

where (for $r = 1, \dots, k$)

$$\mathfrak{Q}(\alpha_r, r) = \left(\begin{array}{c|c|c} I_{r,r} & 0_{r,1} & 0_{r,k-r} \\ \hline q_{1,r} & h^{-\alpha_r} & 0_{1,r-l} \\ \hline 0_{k-r,r} & 0_{k-r,1} & I_{k-r,k-r} \end{array}\right),$$

$$q_{1,r} = \left(\frac{-h^{\alpha_0}}{h^{\alpha_r}} w_{-\alpha_0,r}, \frac{-h^{\alpha_1}}{h^{\alpha_r}} w_{-\alpha_1,r-1}, \dots, \frac{-h^{\alpha_{r-1}}}{h^{\alpha_r}} w_{-\alpha_{r-1},1}\right)$$
$$= \left(\frac{-c_{-\alpha_0,r}}{h^{\alpha_r}}, \frac{-c_{-\alpha_1,r-1}}{h^{\alpha_r}}, \dots, \frac{-c_{-\alpha_{r-1},1}}{h^{\alpha_r}}\right),$$

and $w_{\alpha,i} = (-1)^i \binom{\alpha}{i}$.

Proof. It is obtained after consecutive evaluating Definition 6.2 for $l = 0, 1, \dots, k$. First, for $l = 0$, it can be written as

$$\begin{pmatrix} {}_0^{\mathcal{E}}\Delta_0^{\alpha_0} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_k \end{pmatrix} = \underbrace{\begin{pmatrix} h^{-\alpha_0} & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}}_{\mathfrak{Q}(\alpha_0,0)} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_k \end{pmatrix}.$$

Next, for $l = 1$:

$$\begin{pmatrix} {}_0^{\mathcal{E}}\Delta_0^{\alpha_0} f_0 \\ {}_0^{\mathcal{E}}\Delta_1^{\alpha_1} f_1 \\ f_2 \\ \vdots \\ f_k \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ \frac{-c_{-\alpha_0,1}}{h^{\alpha_1}} & h^{-\alpha_1} & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}}_{\mathfrak{Q}(\alpha_1,1)} \begin{pmatrix} {}_0^{\mathcal{E}}\Delta_0^{\alpha_0} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_k \end{pmatrix},$$

for $l = 2$:

$$\begin{pmatrix} {}_0^{\mathcal{E}}\Delta_0^{\alpha_0} f_0 \\ {}_0^{\mathcal{E}}\Delta_1^{\alpha_1} f_1 \\ {}_0^{\mathcal{E}}\Delta_2^{\alpha_2} f_2 \\ f_3 \\ \vdots \\ f_k \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \frac{-c_{-\alpha_0,2}}{h^{\alpha_2}} & \frac{-c_{-\alpha_1,1}}{h^{\alpha_2}} & h^{-\alpha_2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}}_{\mathfrak{Q}(\alpha_2,2)} \begin{pmatrix} {}_0^{\mathcal{E}}\Delta_0^{\alpha_0} f_0 \\ {}_0^{\mathcal{E}}\Delta_1^{\alpha_1} f_1 \\ f_2 \\ \vdots \\ f_k \end{pmatrix},$$

and generally, for $l = r$, we have

$$\begin{pmatrix} {}_0^{\mathcal{E}}\Delta_0^{\alpha_0} f_0 \\ {}_0^{\mathcal{E}}\Delta_1^{\alpha_1} f_1 \\ \vdots \\ {}_0^{\mathcal{E}}\Delta_{r-1}^{\alpha_{r-1}} f_{r-1} \\ {}_0^{\mathcal{E}}\Delta_r^{\alpha_r} f_r \\ f_{r+1} \\ \vdots \\ f_k \end{pmatrix} = \underbrace{\left(\begin{array}{c|c|c} I_{r,r} & 0_{r,1} & 0_{r,k-r} \\ \hline q_{1,r} & h^{-\alpha_r} & 0_{1,k-r} \\ \hline 0_{k-r,r} & 0_{k-r,1} & I_{k-r,k-r} \end{array}\right)}_{\mathfrak{Q}(\alpha_r, r)} \begin{pmatrix} {}_0^{\mathcal{E}}\Delta_0^{\alpha_0} f_0 \\ {}_0^{\mathcal{E}}\Delta_1^{\alpha_1} f_1 \\ \vdots \\ {}_0^{\mathcal{E}}\Delta_{r-1}^{\alpha_{r-1}} f_{r-1} \\ f_r \\ \vdots \\ f_k \end{pmatrix}.$$

Finally, for $l = k$:

$$\begin{pmatrix} {}_0^{\mathcal{E}}\Delta_0^{\alpha_0} f_0 \\ {}_0^{\mathcal{E}}\Delta_1^{\alpha_1} f_1 \\ \vdots \\ {}_0^{\mathcal{E}}\Delta_{k-1}^{\alpha_{k-1}} f_{k-1} \\ {}_0^{\mathcal{E}}\Delta_k^{\alpha_k} f_k \end{pmatrix} = \underbrace{\left(\begin{array}{c|c} I_{k,k} & 0_{k,1} \\ \hline q_{1,k} & h^{-\alpha_k} \end{array}\right)}_{\mathfrak{Q}(\alpha_k, k)} \begin{pmatrix} {}_0^{\mathcal{E}}\Delta_0^{\alpha_0} f_0 \\ {}_0^{\mathcal{E}}\Delta_1^{\alpha_1} f_1 \\ \vdots \\ {}_0^{\mathcal{E}}\Delta_{k-1}^{\alpha_{k-1}} f_{k-1} \\ f_k \end{pmatrix},$$

where

$$\begin{aligned} q_{1,k} &= \left(\frac{-h^{\alpha_0}}{h^{\alpha_k}} w_{-\alpha_0,k}, \frac{-h^{\alpha_1}}{h^{\alpha_k}} w_{-\alpha_1,k-1}, \ldots, \frac{-h^{\alpha_{k-1}}}{h^{\alpha_r}} w_{-\alpha_{k-1},1} \right) \\ &= \left(\frac{-c_{-\alpha_0,k}}{h^{\alpha_k}}, \frac{-c_{-\alpha_1,k-1}}{h^{\alpha_k}}, \ldots, \frac{-c_{-\alpha_{k-1},1}}{h^{\alpha_k}} \right). \end{aligned}$$

Combining all this together, ends the proof. ■

Remark 6.4. Taking the limit $h \to 0$ to the above considerations, the following matrix form of $\mathcal{E}$-type derivative is given:

$$\begin{pmatrix} {}_0^{\mathcal{E}}\mathrm{D}_0^{\alpha(t)} f(0) \\ {}_0^{\mathcal{E}}\mathrm{D}_h^{\alpha(t)} f(h) \\ {}_0^{\mathcal{E}}\mathrm{D}_{2h}^{\alpha(t)} f(2h) \\ \vdots \\ {}_0^{\mathcal{E}}\mathrm{D}_{kh}^{\alpha(t)} f(kh) \end{pmatrix} = \lim_{h \to 0} \prod_{j=0}^{k} \mathfrak{Q}(\alpha_{k-j}, k-j) \begin{pmatrix} f(0) \\ f(h) \\ f(2h) \\ \vdots \\ f(kh) \end{pmatrix}. \tag{6.2}$$

6.3 Numerical scheme for the $\mathcal{E}$-type operator

This section contains the block diagrams of simple switching case, its generalization for multiple-switching (variable-order) case and corresponding to them numerical schemes equivalent to $\mathcal{E}$-type operator.

Simple output-additive switching order case

The idea of output-additive switching scheme is depicted in Fig. 6.2, where all the switches S_i, $i = 1, 2$, change their positions depending on an actual value of $\alpha(t)$. If we want to switch from α_1 to α_2, then, before switching time T, all switches (S_1 and S_2) are connected to terminals a, and after this time switches change positions to terminals b. At the instant time T, the derivative block of complementary order $\bar{\alpha}_2$ is connected at the end of the current derivative block of order α_1, where

$$\bar{\alpha}_2 = \alpha_2 - \alpha_1. \tag{6.3}$$

If $\bar{\alpha}_2 < 0$, then ${}_T\mathrm{D}_t^{\bar{\alpha}_2}$ corresponds to integration of $f(t)$; and, if $\bar{\alpha}_2 > 0$, then ${}_T\mathrm{D}_t^{\bar{\alpha}_2}$ corresponds to derivative of $f(t)$, with appropriate order $\bar{\alpha}_2$.

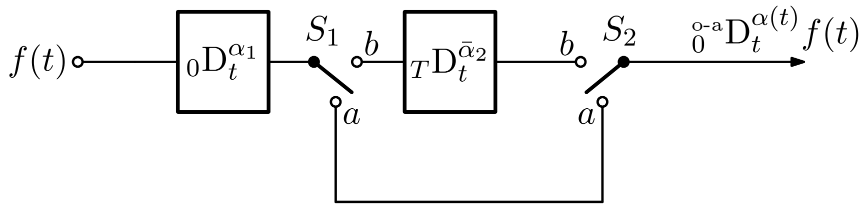


Fig. 6.2. Structure of simple-switching order derivative (switching from α_1 to $\alpha_1 + \bar{\alpha}_2$) as an output-additive switching scheme

Now, the numerical scheme corresponding to the above derivative switching structure is introduced.

Lemma 6.5. *For a switching order case, when the switch from order α_1 to order α_2 occurs at time T (the corresponding sample number $\hat{k}$ is such that $T = \hat{k}h$), the numerical scheme has the following form:*

$$\begin{pmatrix} {}^{\text{o-a}}_{0}\mathrm{D}_0^{\alpha(t)} f(t) \\ {}^{\text{o-a}}_{0}\mathrm{D}_h^{\alpha(t)} f(t) \\ \vdots \\ {}^{\text{o-a}}_{0}\mathrm{D}_{(\hat{k}-1)h}^{\alpha(t)} f(t) \\ {}^{\text{o-a}}_{0}\mathrm{D}_T^{\alpha(t)} f(t) \\ \vdots \\ {}^{\text{o-a}}_{0}\mathrm{D}_{kh}^{\alpha(t)} f(t) \end{pmatrix} = \lim_{h\to 0} W(\bar{\alpha}_2, k, \hat{k}) W(\alpha_1, k) \begin{pmatrix} f(0) \\ f(h) \\ \vdots \\ f(\hat{k}h - h) \\ f(T) \\ \vdots \\ f(kh) \end{pmatrix}, \tag{6.4}$$

where

$$W(\bar{\alpha}_2, k, \hat{k}) = \begin{pmatrix} I_{\hat{k},\hat{k}} & 0_{\hat{k},k-\hat{k}+1} \\ 0_{k-\hat{k}+1,\hat{k}} & W(\bar{\alpha}_2, k-\hat{k}) \end{pmatrix},$$

and

$$\alpha(t) = \begin{cases} \alpha_1 & \text{for } t < T, \\ \alpha_1 + \bar{\alpha}_2 & \text{for } t \geq T. \end{cases}$$

The order $\bar{\alpha}_2$, appearing above, is given by relation (6.3).

Proof. Beginning with time step T, the derivative of order $\bar{\alpha}_2$ is added in series to block of derivative of order α_1. So, the signal incoming to the block of derivative $\bar{\alpha}_2$ is a derivative of order α_1. Thus, the output signal can be described by

$$\begin{pmatrix} {}^{\text{o-a}}_{0}\mathrm{D}^{\alpha(t)}_{0} f(t) \\ {}^{\text{o-a}}_{0}\mathrm{D}^{\alpha(t)}_{h} f(t) \\ \vdots \\ {}^{\text{o-a}}_{0}\mathrm{D}^{\alpha(t)}_{(\hat{k}-1)h} f(t) \\ {}^{\text{o-a}}_{0}\mathrm{D}^{\alpha(t)}_{T} f(t) \\ \vdots \\ {}^{\text{o-a}}_{0}\mathrm{D}^{\alpha(t)}_{kh} f(t) \end{pmatrix} = \lim_{h\to 0} W(\bar{\alpha}_2, k, \hat{k}) \begin{pmatrix} {}_{0}\mathrm{D}^{\alpha_1}_{0} f(t) \\ {}_{0}\mathrm{D}^{\alpha_1}_{h} f(t) \\ \vdots \\ {}_{0}\mathrm{D}^{\alpha_1}_{(\hat{k}-1)h} f(t) \\ {}_{0}\mathrm{D}^{\alpha_1}_{T} f(t) \\ \vdots \\ {}_{0}\mathrm{D}^{\alpha_1}_{kh} f(t) \end{pmatrix} \tag{6.5}$$

The (6.5) written in one equation require an identity sub-matrix corresponding to α_2 block, responsible for passing the input signal directly to the block of α_1 order derivative, until first time interval. Actually, this fact ends the proof. ■

Multiple output-additive switching (variable-order) case

In general case, when there are many switchings between arbitrary orders, we have the following structure presented in Fig. 6.3.

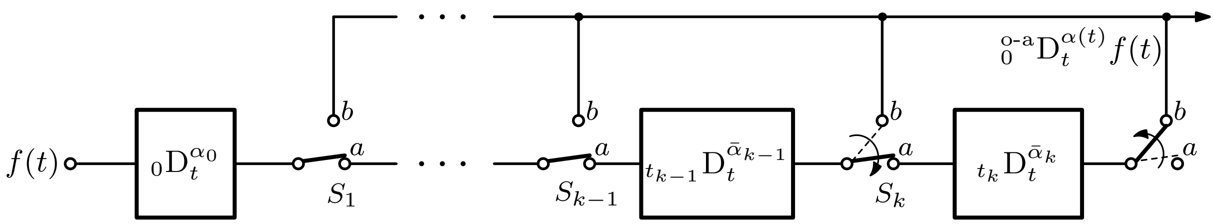


Fig. 6.3. Realization of $\mathcal{E}$-type derivative in the form of switching scheme, where $\bar{\alpha}_j = \alpha_j - \alpha_{j-1}$, $j = 1, \ldots, k$, (configuration at time $t = t_k$)

Let us assume that the order of derivative changes with every time step (which gives a variable-order derivative), i.e.,

$$\alpha(t) = \alpha_j \quad \text{for} \quad jh \leq t \leq (j+1)h, \quad j = 0, \ldots, k,$$

where

$$\alpha_j = \sum_{i=0}^{j} \bar{\alpha}_i, \tag{6.6}$$

where, in this case, $\alpha_0 = \bar{\alpha}_0$ is a value of initial order. Using Lemma 6.5, the following numerical scheme of Fig. 6.3 is obtained

$$\begin{pmatrix} {}^{\text{o-a}}_{0}\mathrm{D}^{\alpha(t)}_{0} f(t) \\ \vdots \\ {}^{\text{o-a}}_{0}\mathrm{D}^{\alpha(t)}_{kh} f(t) \end{pmatrix} = \prod_{j=0}^{k} W(\bar{\alpha}_{k-j}, k, k-j) \begin{pmatrix} f(0) \\ \vdots \\ f(kh) \end{pmatrix}.$$

It can be proved that, the output-additive switching order scheme shown in Fig. 6.3 is equivalent to the $\mathcal{E}$-type fractional variable-order derivative. To do this, let us first introduce Lemma 6.6 and Lemma 6.7.

Lemma 6.6. *The product of matrix forms equivalent to the $\mathcal{B}$- and $\mathcal{E}$-type operators, for negative value of orders gives an identity matrix*

$$ {}^{\mathcal{B}}W(-\alpha, k) \prod_{j=0}^{k} \mathfrak{Q}(\alpha_{k-j}, k-j) = I_{k+1,k+1}. \tag{6.7}$$

Proof. The matrix form of the $\mathcal{B}$-type operator for negative value of orders can be expressed by

$$ {}^{\mathcal{B}}W(-\alpha, k) = \begin{pmatrix} h^{\alpha_0} & 0 & 0 & \dots & 0 & 0 \\ h^{\alpha_0} w_{-\alpha_0,1} & h^{\alpha_1} & 0 & \dots & 0 & 0 \\ h^{\alpha_0} w_{-\alpha_0,2} & h^{\alpha_1} w_{-\alpha_1 1} & h^{\alpha_2} & \dots & 0 & 0 \\ h^{\alpha_0} w_{-\alpha_0,3} & h^{\alpha_1} w_{-\alpha_1,2} & h^{\alpha_2} w_{-\alpha_2,1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ h^{\alpha_0} w_{-\alpha_0,k-1} & h^{\alpha_1} w_{-\alpha_1,k-2} & h^{\alpha_2} w_{-\alpha_2,k-3} & \dots & h^{\alpha_{k-1}} & 0 \\ h^{\alpha_0} w_{-\alpha_0,k} & h^{\alpha_1} w_{-\alpha_1,k-1} & h^{\alpha_2} w_{-\alpha_2,k-2} & \dots & h^{\alpha_{k-1}} w_{-\alpha_{k-1},1} & h^{\alpha_k} \end{pmatrix}.$$

The first matrix product can be described as the following block matrices:

$$\begin{aligned} {}^{\mathcal{B}}W(-\alpha, k)\mathfrak{Q}(\alpha_k, k) &= \left(\begin{array}{c|c} {}^{\mathcal{B}}W(-\alpha, k-1) & 0_{k,1} \\ \hline R(-\alpha, k) & h^{\alpha_k} \end{array}\right) \left(\begin{array}{c|c} I_{k,k} & 0_{k,1} \\ \hline q_{1,k} & h^{-\alpha_k} \end{array}\right) \\ &= \left(\begin{array}{c|c} {}^{\mathcal{B}}W(-\alpha, k-1) & 0_{k,1} \\ \hline 0_{1,k} & 1 \end{array}\right), \end{aligned}$$

where

$$\begin{aligned} R(-\alpha, i) &= \begin{pmatrix} h^{\alpha_0} w_{-\alpha_0,i} & h^{\alpha_1} w_{-\alpha_1,i-1} & \cdots & h^{\alpha_{i-1}} w_{-\alpha_{i-1},1} \end{pmatrix}, \\ h^{\alpha_i} q_{1,i} &= \left(-h^{\alpha_0} w_{-\alpha_0,i}, -h^{\alpha_1} w_{-\alpha_1,i-1}, \dots, -h^{\alpha_{i-1}} w_{-\alpha_{i-1},1}\right). \end{aligned}$$

The second matrix product can be expressed by

$$
\begin{aligned}
&{}^{\mathcal{B}}W(-\alpha,k)\mathfrak{Q}(\alpha_k,k)\mathfrak{Q}(\alpha_{k-1},k-1)\\
&=\left(\begin{array}{cc|c} {}^{\mathcal{B}}W(-\alpha,k-2) & 0_{k-1,1} & 0_{k-1,1}\\ R(-\alpha,k-1) & h^{\alpha_{k-1}} & 0\\ \hline 0_{1,k-1} & 0 & 1\end{array}\right)\left(\begin{array}{cc|c} I_{k-1,k-1} & 0_{k-1,1} & 0_{k-1,1}\\ q_{1,k-1} & h^{-\alpha_{k-1}} & 0\\ \hline 0_{1,k-1} & 0 & 1\end{array}\right)\\
&=\left(\begin{array}{cc|c} {}^{\mathcal{B}}W(-\alpha,k-2) & 0_{k-1,1} & 0_{k-1,1}\\ 0_{1,k-1} & 1 & 0\\ \hline 0_{1,k-1} & 0 & 1\end{array}\right).
\end{aligned}
$$

For general case, when $j=r$ we have the following matrix product

$$
\begin{aligned}
&\left({}^{\mathcal{B}}W(-\alpha,k)\prod_{j=0}^{r-1}\mathfrak{Q}(\alpha_{k-j},k-j)\right)\mathfrak{Q}(\alpha_{k-r},k-r)\\
&=\left(\begin{array}{cc|c} {}^{\mathcal{B}}W(-\alpha,k-r-1) & 0_{k-r,1} & 0_{k-r,r}\\ R(-\alpha,k-r) & h^{\alpha_{k-r}} & 0\\ \hline 0_{r,k-r} & 0_{r,1} & I_{r,r}\end{array}\right)\left(\begin{array}{cc|c} I_{k-r,k-r} & 0_{k-r,1} & 0_{k-r,r}\\ q_{1,k-r} & h^{-\alpha_{k-r}} & 0_{1,r}\\ \hline 0_{r,k-r} & 0_{r,1} & I_{r,r}\end{array}\right)\\
&=\left(\begin{array}{cc|c} {}^{\mathcal{B}}W(-\alpha,k-r-1) & 0_{k-r,1} & 0_{k-r,r}\\ 0_{1,k-r} & 1 & 0\\ \hline 0_{r,k-r} & 0_{r,1} & I_{r,r}\end{array}\right).
\end{aligned}
$$

Finally, for $j=k$ matrix product equals to

$$
\begin{aligned}
\left({}^{\mathcal{B}}W(-\alpha,k)\prod_{j=0}^{k-1}\mathfrak{Q}(\alpha_{k-j},k-j)\right)\mathfrak{Q}(\alpha_0,0)&=\left(\begin{array}{c|c} h^{\alpha_0} & 0_{1,k}\\ \hline 0_{k,1} & I_{k,k}\end{array}\right)\left(\begin{array}{c|c} h^{-\alpha_0} & 0_{1,k}\\ \hline 0_{k,1} & I_{k,k}\end{array}\right)\\
&=\left(\begin{array}{c|c} 1 & 0_{1,k}\\ \hline 0_{k,1} & I_{k,k}\end{array}\right),
\end{aligned}
$$

Consecutive multiplying all of terms from Lemma 6.6 ends the proof. ∎

Lemma 6.7. *The product of multi-switching numerical scheme equivalent to the $\mathcal{B}$-type derivative and the multi-switching output-additive scheme gives an identity matrix, i.e.,*

$$
\underbrace{\left(\prod_{j=0}^{k}W(-\bar{\alpha}_j,k,j)\right)}_{{}^{\mathcal{B}}W(-\alpha,k)}\left(\prod_{j=0}^{k}W(\bar{\alpha}_{k-j},k,k-j)\right)=I_{k+1,k+1}.
$$

Proof. By extension of above terms and using connectivity law the following holds

$$\prod_{j=0}^{k-1} W(-\bar{\alpha}_j, k, j) \underbrace{W(-\bar{\alpha}_k, k, k) W(\bar{\alpha}_k, k, k)}_{I_{k+1,k+1}} \prod_{j=1}^{k} W(\bar{\alpha}_{k-j}, k, k-j) =$$

$$\prod_{j=0}^{k-2} W(-\bar{\alpha}_j, k, j) \underbrace{W(-\bar{\alpha}_{k-1}, k, k-1) W(\bar{\alpha}_{k-1}, k, k-1)}_{I_{k+1,k+1}} \prod_{j=2}^{k} W(\bar{\alpha}_{k-j}, k, k-j) =$$

$$\prod_{j=0}^{k-l+1} W(-\bar{\alpha}_j, k, j) \underbrace{W(-\bar{\alpha}_{k-l}, k, k-l) W(\bar{\alpha}_{k-l}, k, k-l)}_{I_{k+1,k+1}} \prod_{j=l-1}^{k} W(\bar{\alpha}_{k-j}, k, k-j) =$$

$$\cdots = I_{k+1,k+1}.$$

■

Finally, the following theorem can be formulated:

Theorem 6.8. *The output-additive switching order scheme presented in Fig. 6.3 is equivalent to the $\mathcal{E}$-type variable-order derivative given by Definition 6.1.*

Proof. The proof is a conclusion directly based on Lemma 6.6 and Lemma 6.7.

$$^{\mathcal{B}}W(-\alpha, k) \prod_{j=0}^{k} \mathfrak{Q}(\alpha_{k-j}, k-j) = {}^{\mathcal{B}}W(-\alpha, k) \prod_{j=0}^{k} W(\bar{\alpha}_{k-j}, k, k-j),$$

thus,

$$\prod_{j=0}^{k} \mathfrak{Q}(\alpha_{k-j}, k-j) = \prod_{j=0}^{k} W(\bar{\alpha}_{k-j}, k, k-j),$$

where $\alpha_j = \sum_{i=0}^{j} \bar{\alpha}_i$. ■

Above considerations show that composition of $\mathcal{B}$- and $\mathcal{E}$-type fractional variable-order operators for negative values of orders give an identity. So, Lemma 6.7 and Lemma 6.6 imply the duality property between $\mathcal{B}$- and $\mathcal{E}$-type operators.

Remark 6.9. [78] The duality property between $\mathcal{B}$ and $\mathcal{E}$-types variable-order operators

$${}^{\mathcal{B}}_{0}\mathrm{D}_t^{\alpha(t)} \left({}^{\mathcal{E}}_{0}\mathrm{D}_t^{-\alpha(t)} f(t) \right) = {}^{\mathcal{E}}_{0}\mathrm{D}_t^{\alpha(t)} \left({}^{\mathcal{B}}_{0}\mathrm{D}_t^{-\alpha(t)} f(t) \right) = f(t).$$

6.4 Initial conditions of the $\mathcal{E}$-type operator

This section contains the numerical scheme for considering the initial conditions into the $\mathcal{E}$-type fractional variable-order definition in the form of finite and infinite length. It is some kind of reference to initial conditions for recursive constant (Sections 2.4) and $\mathcal{D}$-type operators (Section 5.4). The $\mathcal{E}$-type recursive difference for time k^* is defined by [29]:

$$ {}^{\mathcal{E}}_{0}\Delta^{\alpha_{k^*}}_{k^*} x_{k^*} = \frac{x_{k^*}}{h^{\alpha_{k^*}}} - \sum_{j=1}^{k^*} (-1)^j \binom{-\alpha_{k^*-j}}{j} \frac{h^{\alpha_{k^*-j}}}{h^{\alpha^*_k}} {}^{\mathcal{E}}_{0}\Delta^{\alpha_{k^*-j}}_{k^*-j} x_{k^*-j} \, . \tag{6.8} $$

Obviously, (6.8) can be interpreted as a situatin where zero time is shifted to the point $t^* = T$. And, it means, that the system possesses some initial energy (history). In this case, the difference can be rewritten in the following form for $k = k^* - T$ [29]:

$$ {}^{\mathcal{E}}_{-T}\Delta^{\alpha_k}_{k} x_k = \frac{x_k}{h^{\alpha_k}} - \sum_{j=1}^{k+T} (-1)^j \binom{-\alpha_{k-j}}{j} \frac{h^{\alpha_{k-j}}}{h^{\alpha_k}} {}^{\mathcal{E}}_{-T}\Delta^{\alpha_{k-j}}_{k-j} x_{k-j} \, . \tag{6.9} $$

Thus, (6.9) can be rewritten to the form [29]

$$ \begin{aligned} {}^{\mathcal{E}}_{-T}\Delta^{\alpha_k}_{k} x_k \;&=\; \frac{x_k}{h^{\alpha_k}} - \sum_{j=1}^{k-1} (-1)^j \binom{-\alpha_{k-j}}{j} \frac{h^{\alpha_{k-j}}}{h^{\alpha_k}} {}^{\mathcal{E}}_{-T}\Delta^{\alpha_{k-j}}_{k-j} x_{k-j} \\ &-\; \sum_{j=k}^{k+T} (-1)^j \binom{-\alpha_{k-j}}{j} \frac{h^{\alpha_{k-j}}}{h^{\alpha_k}} {}^{\mathcal{E}}_{-T}\Delta^{\alpha_{k-j}}_{k-j} x_{k-j} \, . \end{aligned} \tag{6.10} $$

By assumption that $\alpha_{k-j} = \alpha_0$ (where α_0=const) for $j \geq k$ we can write (6.10) in the form [29]

$$ \begin{aligned} {}^{\mathcal{E}}_{-T}\Delta^{\alpha_k}_{k} x_k \;&=\; \frac{x_k}{h^{\alpha_k}} - \sum_{j=1}^{k-1} (-1)^j \binom{-\alpha_{k-j}}{j} \frac{h^{\alpha_{k-j}}}{h^{\alpha_k}} {}^{\mathcal{E}}_{-T}\Delta^{\alpha_{k-j}}_{k-j} x_{k-j} \\ &-\; \frac{h^{\alpha_0}}{h^{\alpha_k}} \sum_{j=k}^{k+T} (-1)^j \binom{-\alpha_0}{j} {}^{\mathcal{E}}_{-T}\Delta^{\alpha_0}_{k-j} x_{k-j} \, . \end{aligned} \tag{6.11} $$

The condition $\alpha_{k-j} = \alpha_0$ (for $j \geq k$) means that until zero-time all past α_{k-j} orders equals to α_0 constant-order. The fractional order differences from ${}^{\mathcal{E}}\Delta^{\alpha_0} x_0$ to ${}^{\mathcal{E}}\Delta^{\alpha_0} x_{-T}$ can be admitted as initial conditions statement for $\mathcal{E}$-type fractional order difference. The initial condition case is interpreted as a series from ${}^{\mathcal{E}}\Delta^{\alpha_0} x_0$ to ${}^{\mathcal{E}}\Delta^{\alpha_0} x_{-T}$.

Constant, finite-time initial conditions case

When the system is in a steady state before for example control algorithm starts ($t < 0$), what is easy to imagine in practical application, we can assume that ${}^{\mathcal{E}}_{-T}\Delta^{\alpha_0}_{i} x_i = c = \text{const}$

for $i = -T \dots 0$. In this can we can obtain [29]

$$ {}^{\mathcal{E}}_{-T}\Delta^{\alpha_k}_{k} x_k = \frac{x_k}{h^{\alpha_k}} - \sum_{j=1}^{k-1} (-1)^j \binom{-\alpha_{k-j}}{j} \frac{h^{\alpha_{k-j}}}{h^{\alpha_k}} {}^{\mathcal{E}}_{-T}\Delta^{\alpha_{k-j}}_{k-j} x_{k-j} - \frac{h^{\alpha_0}}{h^{\alpha_k}} \sum_{j=k}^{k+T} (-1)^j \binom{-\alpha_0}{j} c \,. \tag{6.12} $$

Constant, infinite-time initial conditions case

For infinite length of initial conditions, (6.12) can be rewritten to the form of the following theorem:

Theorem 6.10. *[29] The $\mathcal{E}$-type fractional order difference for initial conditions in the form of infinite length constant function ${}^{\mathcal{E}}\Delta^{\alpha_0} x_i = c = const$ and $\alpha_0 = const$ for $i = -\infty \dots 0$ is given by the following relation:*

$$ {}^{\mathcal{E}}_{-\infty}\Delta^{\alpha_k}_{k} x_k = \frac{x_k}{h^{\alpha_k}} - \sum_{j=1}^{k-1} (-1)^j \left[\binom{-\alpha_{k-j}}{j} \frac{h^{\alpha_{k-j}}}{h^{\alpha_k}} {}^{\mathcal{E}}_{-\infty}\Delta^{\alpha_{k-j}}_{k-j} x_{k-j} - \binom{-\alpha_0}{j} \frac{h^{\alpha_0}}{h^{\alpha_k}} c \right] + \frac{h^{\alpha_0}}{h^{\alpha_k}} c \,. \tag{6.13} $$

Proof. Due to the α_0 is a constant value, the second sum in (6.11), for infinite length of initial conditions $T \to \infty$ (it implies that also $T + k \to \infty$), can be obtained in the following form

$$ \sum_{j=0}^{\infty} (-1)^j \binom{-\alpha_0}{j} c = \sum_{j=0}^{k-1} (-1)^j \binom{-\alpha_0}{j} c + \sum_{j=k}^{k+T} (-1)^j \binom{-\alpha_0}{j} c \,. $$

Using Remark 2.10 we obtain the following

$$ \begin{aligned} \sum_{j=k}^{k+T} (-1)^j \binom{-\alpha_0}{j} c &= \sum_{j=0}^{\infty} (-1)^j \binom{-\alpha_0}{j} c - \sum_{j=0}^{k-1} (-1)^j \binom{-\alpha_0}{j} c \\ &= -c - \sum_{j=1}^{k-1} (-1)^j \binom{-\alpha_0}{j} c \,, \end{aligned} $$

and combining with (6.12) end the proof. ∎

Continuous-time case

In the continuous-time domain the scheme for including the initial conditions in the form of infinite length can be introduced as:

Theorem 6.11. *[29] The $\mathcal{E}$-type fractional variable-order derivative for initial conditions in the form of infinite length constant function ${}^{\mathcal{E}}_{-\infty}\mathrm{D}^{\alpha(t)}_{t} f(t) = c = const$ for $t = (-\infty, 0)$ is given by the following relation:*

$$\begin{aligned} {}^{\mathcal{E}}_{-\infty}\mathrm{D}^{\alpha(t)}_{t} f(t) &= \frac{f(t)}{h^{\alpha(t)}} - \sum_{j=1}^{k-1}(-1)^j \left[\binom{-\alpha(t-jh)}{j} \frac{h^{\alpha(t-jh)}}{h^{\alpha(t)}} {}^{\mathcal{E}}_{-\infty}\mathrm{D}^{\alpha(t-jh)}_{t-jh} f(t) \right. \\ &- \left. \binom{-\alpha(0)}{j} \frac{h^{\alpha(0)}}{h^{\alpha(t)}} c \right] + \frac{h^{\alpha(0)}}{h^{\alpha(t)}} c \,. \end{aligned}$$

Proof. The proof can be done in similar way as proof of Theorem 6.10. ■

6.5 Orders composition properties

In this section, the orders composition properties of the $\mathcal{E}$-type derivative with constant-order differ-integral is shown based on output-additive switching scheme. The multiple output-additive switching structure represents the situation, when order of the $\mathcal{E}$-type operator can be changed in every time step. The order composition properties with constant-order differ-integral can be proved based on equivalence between the $\mathcal{E}$-type definition and its switching scheme. Then, it can be proved that the composition property with constant-order differ-integral holds, but only in one direction, i.e., it is not commutative, and it depends on the sequence of composition. Two cases of composition orders are depicted in Fig. 6.4 and Fig. 6.5. Let us consider the switching scheme presented

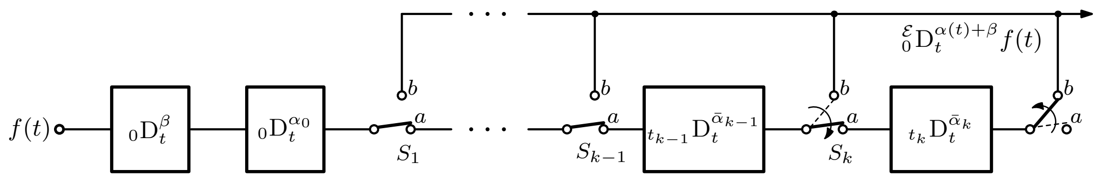


Fig. 6.4. Composition of output-additive switching scheme with fractional constant-order differ-integral from left-hand side, i.e., ${}^{\mathcal{E}}_0\mathrm{D}^{\alpha(t)}_t {}_0\mathrm{D}^{\beta}_t f(t) = {}^{\mathcal{E}}_0\mathrm{D}^{\alpha(t)+\beta}_t f(t)$.

in Fig. 6.4 with fractional constant-order derivative from the left-hand side. Then, the following theorem can be written:

Theorem 6.12. *[32] The following composition property holds*

$${}^{\mathcal{E}}_0\mathrm{D}^{\alpha(t)}_t {}_0\mathrm{D}^{\beta}_t f(t) = {}^{\mathcal{E}}_0\mathrm{D}^{\alpha(t)+\beta}_t f(t),$$

for $\alpha(t) \neq \mathrm{const}$ *and* $\beta \neq 0$.

Proof. Directly based on Fig. 6.4, the following equation can be formulated

$$ {}_{t_k}\mathrm{D}_t^{\bar{\alpha}_k}\cdots{}_{t_1}\mathrm{D}_t^{\bar{\alpha}_1}{}_0\mathrm{D}_t^{\alpha_0}{}_0\mathrm{D}_t^{\beta}f(t)={}_0^{\mathcal{E}}\mathrm{D}_t^{\alpha(t)}{}_0\mathrm{D}_t^{\beta}f(t). \tag{6.14} $$

Due to the composition property for fractional constant-order operator given by (2.3), (6.14) can be rewritten to the form

$$ {}_{t_k}\mathrm{D}_t^{\bar{\alpha}_k}\cdots{}_{t_1}\mathrm{D}_t^{\bar{\alpha}_1}{}_0\mathrm{D}_t^{\alpha_0+\beta}f(t)={}_0^{\mathcal{E}}\mathrm{D}_t^{\alpha(t)+\beta}f(t), $$

which ends the proof. ■

To show that composition property is not commutative it is enough to prove that such property does not holds on time interval for $0<t\le t_2$. Then, we can constraint the output-additive switching scheme presented in Fig. 6.5 to time t_2 and formulate a Theorem 6.14.

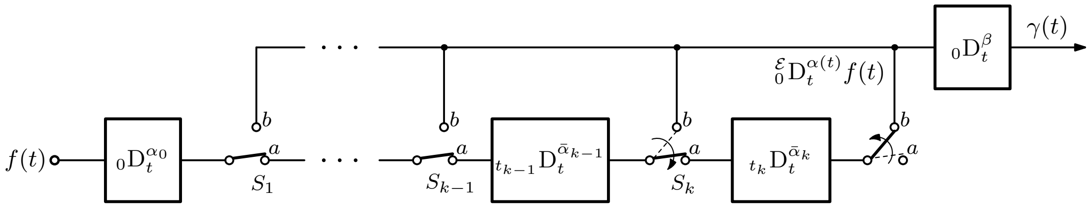


Fig. 6.5. Composition of output-additive switching scheme with fractional constant-order differ-integral from right-hand side, i.e., ${}_0\mathrm{D}_t^{\beta}\left({}_0^{\mathcal{E}}\mathrm{D}_t^{\alpha(t)}f(t)\right)=\gamma(t)\neq{}_0^{\mathcal{E}}\mathrm{D}_t^{\alpha(t)+\beta}f(t)$.

Lemma 6.13. *[32] The fractional constant-order derivative in the Grünwald-Letnikov form given by Def. 2.1 can be expressed by*

$$ \begin{aligned} {}_0\mathrm{D}_t^{\alpha}f(t) &= \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{j=0}^{t}(-1)^j\binom{\alpha}{j}f(t-jh) \\ &= \lim_{h\to 0}\frac{1}{h^{\alpha}}\left(\sum_{j=0}^{\frac{t_1}{h}}(-1)^j\binom{\alpha}{j}f(t-jh)+\sum_{\frac{t_1}{h}+1}^{\frac{t}{h}}(-1)^j\binom{\alpha}{j}f(t-jh)\right) \\ &= \left({}_{t_1}\mathrm{D}_t^{\alpha}+{}_0\mathrm{D}_{t_1-h}^{\alpha}\right)f(t)\implies {}_{t_1}\mathrm{D}_t^{\alpha}f(t)=\left({}_0\mathrm{D}_t^{\alpha}-{}_0\mathrm{D}_{t_1-h}^{\alpha}\right)f(t), \end{aligned} $$

where h is a step time and $0<t_1<t$.

Theorem 6.14. *[32] The composition of output-additive switching scheme with fractional constant-order differ-integral from right-hand side is expressed by*

$$ {}_0\mathrm{D}_{t_2}^{\beta}\underbrace{{}_{t_1}\mathrm{D}_{t_2}^{\alpha_1}{}_0\mathrm{D}_{t_2}^{\alpha_0}f(t)}_{{}_0^{\mathcal{E}}\mathrm{D}_{t_2}^{\alpha(t)}f(t)}=\gamma(t)\neq{}_0^{\mathcal{E}}\mathrm{D}_{t_2}^{\alpha(t)+\beta}f(t), \tag{6.15} $$

for $\alpha(t) \neq \text{const}$ *and* $\beta \neq 0$.

Proof. Let us begin with (6.14), when the composition property holds and time $t = t_2$

$$ {}_{t_1}\mathrm{D}_{t_2}^{\bar{\alpha}_1}\,{}_0\mathrm{D}_{t_2}^{\alpha_0}\,{}_0\mathrm{D}_{t_2}^{\beta} f(t) = {}_0^{\mathcal{E}}\mathrm{D}_{t_2}^{\alpha(t)+\beta} f(t). \tag{6.16} $$

Applying the commutative property of fractional constant-order derivative (2.3) and Lemma 6.13 to (6.16), we can extend this equation to the following form

$$ \begin{aligned} \left({}_0\mathrm{D}_{t_2}^{\bar{\alpha}_1} - {}_0\mathrm{D}_{t_1-h}^{\bar{\alpha}_1}\right){}_0\mathrm{D}_{t_2}^{\beta}\,{}_0\mathrm{D}_{t_2}^{\alpha_0} f(t) &= {}_0\mathrm{D}_{t_2}^{\bar{\alpha}_1}\,{}_0\mathrm{D}_{t_2}^{\beta}\,{}_0\mathrm{D}_{t_2}^{\alpha_0} f(t) - {}_0\mathrm{D}_{t_1-h}^{\bar{\alpha}_1}\,{}_0\mathrm{D}_{t_2}^{\beta}\,{}_0\mathrm{D}_{t_2}^{\alpha_0} f(t) \\ &= {}_0\mathrm{D}_{t_2}^{\beta}\left({}_0\mathrm{D}_{t_1-h}^{\bar{\alpha}_1} + {}_{t_1}\mathrm{D}_{t_2}^{\bar{\alpha}_1}\right){}_0\mathrm{D}_{t_2}^{\alpha_0} f(t) - {}_0\mathrm{D}_{t_1-h}^{\bar{\alpha}_1}\,{}_0\mathrm{D}_{t_2}^{\beta}\,{}_0\mathrm{D}_{t_2}^{\alpha_0} f(t) \\ &= {}_0\mathrm{D}_{t_2}^{\beta}\,{}_{t_1}\mathrm{D}_{t_2}^{\bar{\alpha}_1}\,{}_0\mathrm{D}_{t_2}^{\alpha_0} f(t) + {}_0\mathrm{D}_{t_2}^{\beta}\,{}_0\mathrm{D}_{t_1-h}^{\bar{\alpha}_1}\,{}_0\mathrm{D}_{t_2}^{\alpha_0} f(t) - {}_0\mathrm{D}_{t_1-h}^{\bar{\alpha}_1}\,{}_0\mathrm{D}_{t_2}^{\beta}\,{}_0\mathrm{D}_{t_2}^{\alpha_0} f(t) \\ &= \gamma(t) + {}_0\mathrm{D}_{t_2}^{\beta}\,{}_0\mathrm{D}_{t_1-h}^{\bar{\alpha}_1}\,{}_0\mathrm{D}_{t_2}^{\alpha_0} f(t) - {}_0\mathrm{D}_{t_1-h}^{\bar{\alpha}_1}\,{}_0\mathrm{D}_{t_2}^{\beta}\,{}_0\mathrm{D}_{t_2}^{\alpha_0} f(t). \end{aligned} $$

Comparing above to Theorem 6.14 ends the proof. ■

6.6 Experimental setup

An analog realization of switching system, directly based on $\mathcal{E}$-type fractional variable-order integral, is presented in Fig. 6.6. The fractional integrators are realized using domino-ladder approximations. Due to the use of operational amplifiers, the signals with inverted polarization are obtained. This required amplifiers with gain -1 for each integrator (amplifiers A_2 and A_4). In setup the scheme based on amplifiers A_1 and A_3 contains resistors R and fractional order impedances Z_1, Z_2. As a realization of switches S_1 and S_2, integrated analog switches DG303 are used. The experimental circuit is connected to the dSPACE DS1104 PPC card with a PC. At the beginning, switches S_1 and S_2 are in position a and after order switching, at time instant T, both switches change their positions to terminals b. At first, the experimental setup contains A_1 and A_2 amplifiers and after switching time, the amplifiers A_3 and A_4 are added.

6.6.1 Results of the $\mathcal{E}$-type integrators

Example 6.15. [31] Integrator with order switching from $\alpha = 0.5$ to $\alpha = 1$.

In this case, the scheme presented in Fig. 6.6, contains the following structure: Z_1 and Z_2 are the half-order impedance, $R = 100\text{k}\Omega$. The identification results are obtained by numerical minimization of time responses square error with sampling time 0.001 s and input signal being $u(t) = 0.1 \cdot 1(t)$ V.

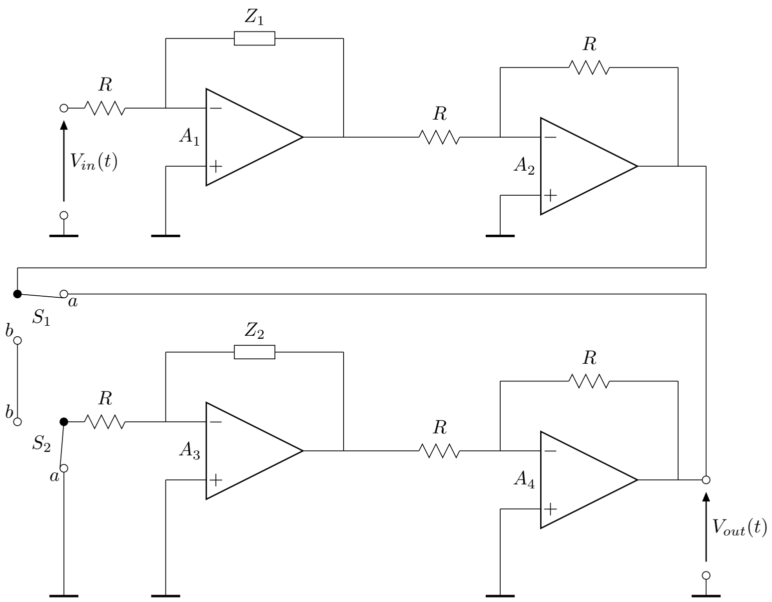


Fig. 6.6. Analog realization of the $\mathcal{E}$-type fractional variable-order integrator.

After identification process the following models for orders -0.5 and -1, in time domain, are obtained:

$$y(t) = {}_0\mathrm{D}_t^{-0.5} a_1 u(t) = 1.35 {}_0\mathrm{D}_t^{-0.5} u(t),$$
$$y(t) = {}_0\mathrm{D}_t^{-1} a_2 u(t) = 1.88 {}_0\mathrm{D}_t^{-1} u(t),$$

which gives rise to the following variable-order integrator:

$$y(t) = {}_0^{\mathcal{E}}\mathrm{D}_t^{-\alpha(t)} \left[a(t)u(t)\right],$$

where (for the switching time $T = 0.1$ s)

$$a(t) = \begin{cases} 1.35 & \text{for } t < 0.1, \\ 1.88 & \text{for } t \geq 0.1, \end{cases}$$

and

$$\alpha(t) = \begin{cases} 0.5 & \text{for } t < 0.1, \\ 1 & \text{for } t \geq 0.1. \end{cases}$$

Identification results and the difference between step responses of analog model and its numerical implementation are presented in Fig. 6.7 for order $\alpha = 0.5$, and in Fig. 6.8 for order $\alpha = 1$.

The experimental results compared to numerical implementation of the $\mathcal{E}$-type variable-order integrator are presented in Fig. 6.9.

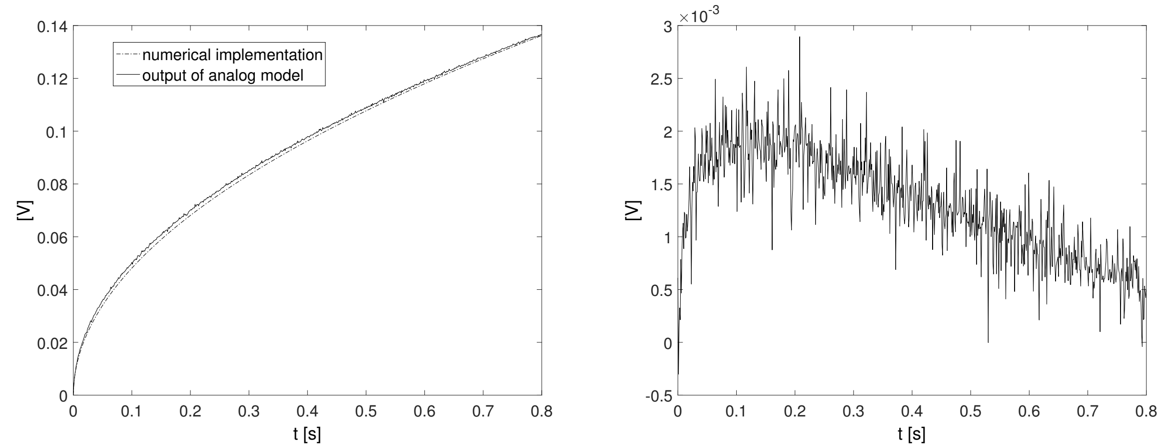


Fig. 6.7. Identification results of constant-order $\alpha = 0.5$ integrator (*left*) and their modeling error (*right*).

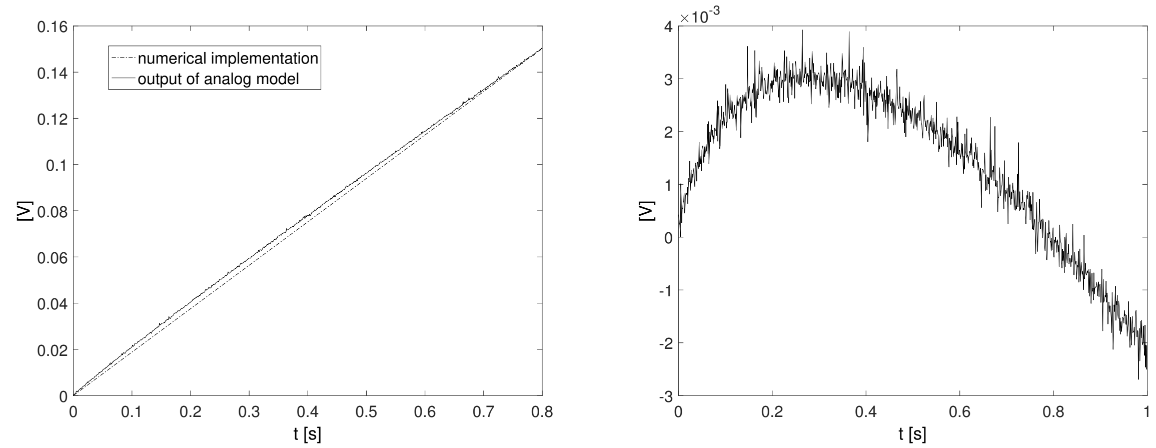


Fig. 6.8. Identification results of constant-order $\alpha = 1$ (*left*) integrator and their modeling error (*right*).

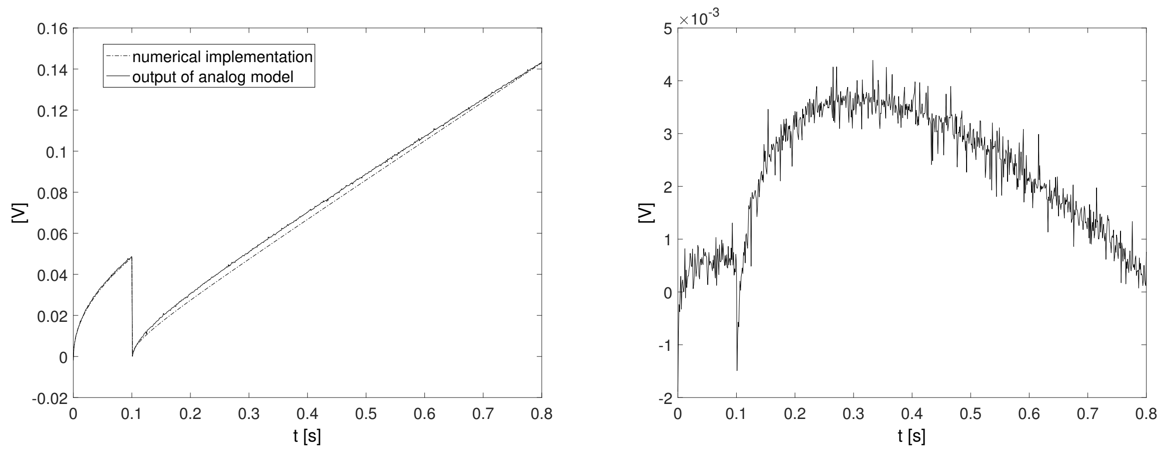


Fig. 6.9. Results of analog and numerical implementation of the $\mathcal{E}$-type integrator with switching order from $\alpha - 0.5$ to $\alpha = 1$ (*left*) and their modeling error (*right*).

Example 6.16. Integrator with order switching from $\alpha = 0.25$ to $\alpha = 0.5$.

In this case, the scheme presented in Fig. 6.6, contains the following structure: Z_1 and Z_2 are the quarter-order impedance, $R = 16\text{k}\Omega$. The identification results are obtained by numerical minimization of time responses square error with sampling time 0.001 s.

After identification process the following models for orders -0.25 and -0.5, in time domain, are obtained:

$$y(t) = {}_0\mathrm{D}_t^{-0.25} a_1 u(t) = 1.44 {}_0\mathrm{D}_t^{-0.25} u(t),$$
$$y(t) = {}_0\mathrm{D}_t^{-0.5} a_2 u(t) = 2.1 {}_0\mathrm{D}_t^{-0.5} u(t),$$

which gives rise to the following variable-order integrator:

$$y(t) = {}_0^{\mathcal{E}}\mathrm{D}_t^{-\alpha(t)} \left[a(t)u(t)\right],$$

where (for the switching time $T = 0.5$ s)

$$a(t) = \begin{cases} 1.44 & \text{for } t < 0.5, \\ 2.1 & \text{for } t \geq 0.5, \end{cases}$$

and

$$\alpha(t) = \begin{cases} 0.25 & \text{for } t < 0.5, \\ 0.5 & \text{for } t \geq 0.5. \end{cases}$$

Identification results and the difference between step responses of analog model and its numerical implementation are presented in Fig. 6.10 for order $\alpha = 0.25$, and in Fig. 6.11 for order $\alpha = 0.5$. Finally, the experimental results compared to numerical implementation

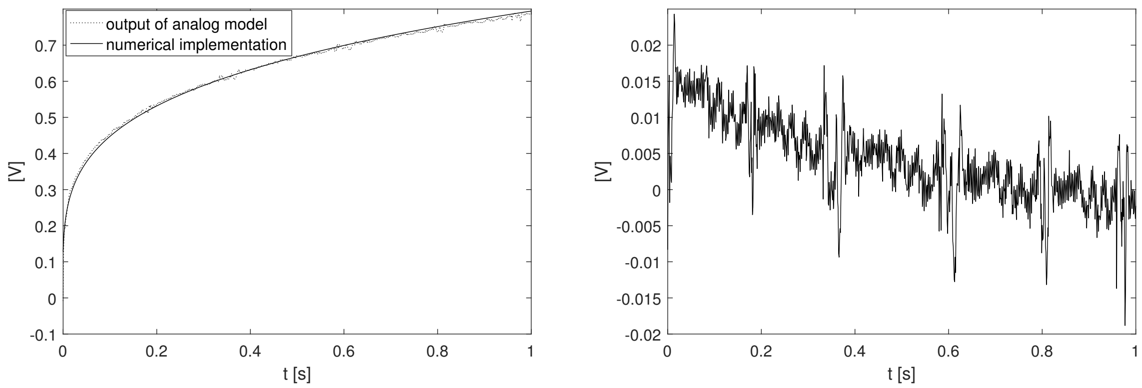


Fig. 6.10. Identification results of constant-order $\alpha = 0.25$ integrator (*left*) and their modeling error (*right*). The input signal equals to $0.5 \cdot 1(t)$ V.

of the $\mathcal{E}$-type variable-order derivative are presented in Fig. 6.12.

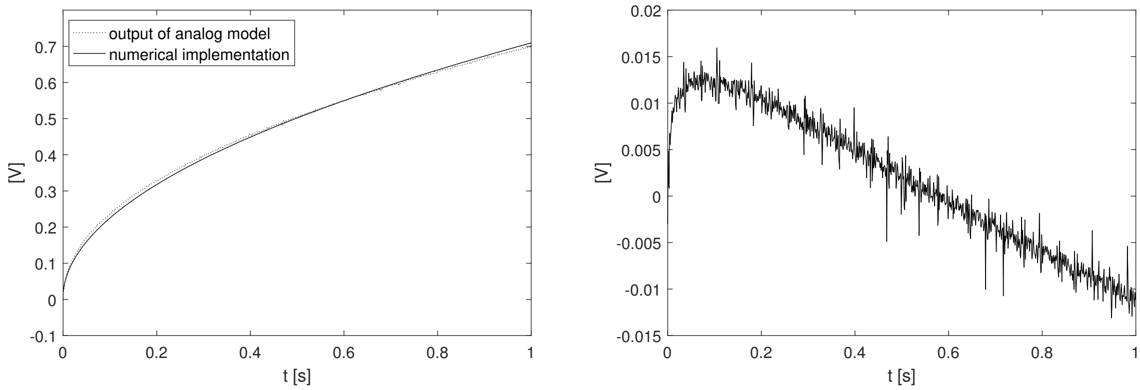


Fig. 6.11. Identification results of constant-order $\alpha = 0.5$ integrator (*left*) and their modeling error (*right*). The input signal equals to $0.3 \cdot 1(t)$ V.

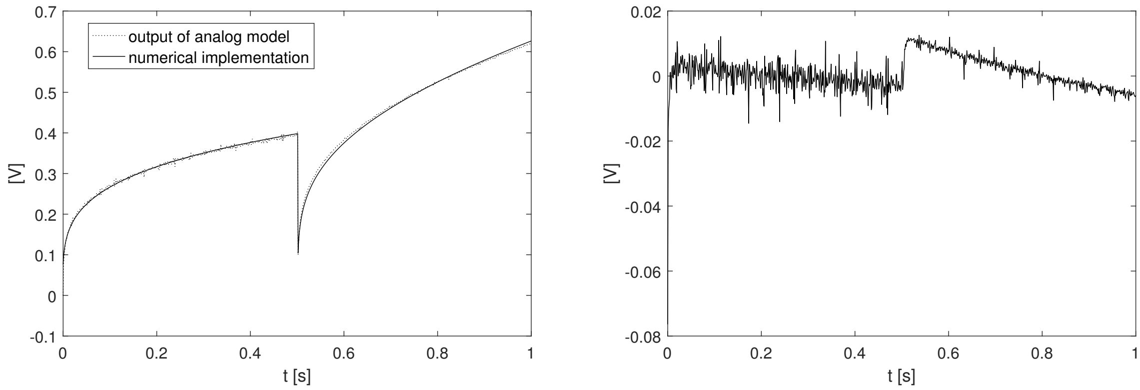


Fig. 6.12. Results of analog and numerical implementation of the $\mathcal{E}$-type integrator with switching order from $\alpha = 0.25$ to $\alpha = 0.5$ (*left*) and their modeling error (*right*). The input signal equals to $0.3 \cdot 1(t)$ V.

6.6.2 Results of the $\mathcal{E}$-type inertial systems

Providing the $\mathcal{E}$-type fractional variable-order integral (Fig. 6.6) into forward path in unity feedback loop shown in Fig. 6.13 there is a possibility to make a realization of $\mathcal{E}$-type fractional variable-order inertial system.

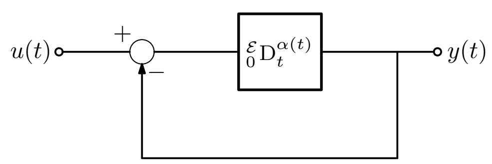


Fig. 6.13. Realization of the $\mathcal{E}$-type fractional variable-order inertial system based on fractional variable-order integral system presented in Fig. 6.6.

Example 6.17. [33] Inertial system with order switching from -0.5 to -1.

In this case the scheme presented in Fig. 6.6 takes the same structure as in Example 6.15. Thus, the parameters' identification of half- and first-order integrals are shown in Fig. 6.7 and Fig 6.8. The identification results are obtained by numerical minimization of time responses square error with sampling time 0.001 s and input signal equals to

$u(t) = 0.5 \cdot 1(t)$ V. Until switching time, the S_1 switch is connected to terminal marked as a, and S_2 switch is connected to terminal b. After switching time ($T \geq 0.1$ s) switches change their positions.

Identified variable-order inertial system has the following form:

$$y(t) = {}_0^{\mathcal{E}}\mathrm{D}_t^{\alpha(t)} \left[a(t)(u(t) - y(t)) \right],$$

where (for the switching time $T = 0.1$ s)

$$a(t) = \begin{cases} 1.35 & \text{for } t < 0.1, \\ 1.88 & \text{for } t \geq 0.1, \end{cases} \qquad \alpha(t) = \begin{cases} -0.5 & \text{for } t < 0.1, \\ -1 & \text{for } t \geq 0.1. \end{cases}$$

The experimental results of fractional variable-order inertial system with order switching between -0.5 and -1, compared to their numerical results, are presented in Fig. 6.14.

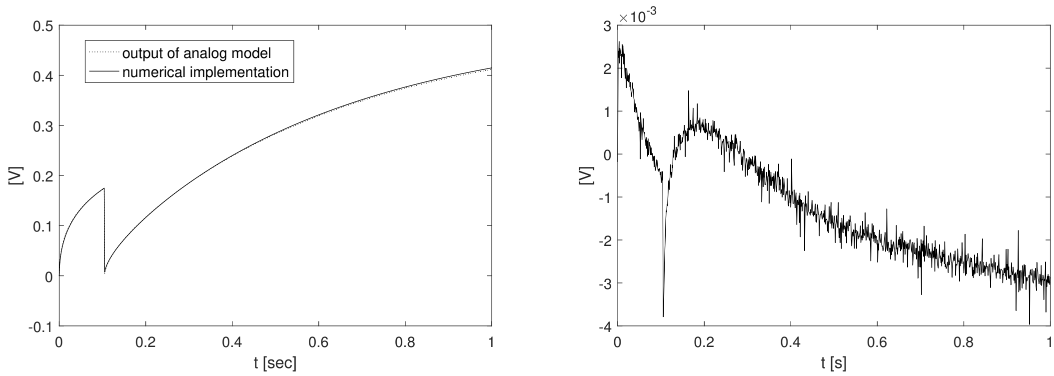


Fig. 6.14. Results of analog and numerical implementation of the $\mathcal{E}$-type inertial system with switching order from -0.5 to -1 (*left*) and their modeling error (*right*).

Example 6.18. Inertial system with order switching from -0.25 to -0.5.

In this case the scheme presented in Fig. 6.6 takes the same structure as in Example 6.16. Thus, the parameters' identification of quarter and half order integrals are shown in Fig. 6.10 and Fig 6.11. The identification results are obtained by numerical minimization of time responses square error with sampling time 0.001 s and input signal being $u(t) = 0.5 \cdot 1(t)$ V. Until switching time the S_1 switch is connected to terminal marked as a, and S_2 switch is connected to terminal b. After switching time ($T \geq 0.5$ s) switches change their positions.

Identified variable-order inertial system has the following form:

$$y(t) = {}_0^{\mathcal{E}}\mathrm{D}_t^{\alpha(t)} \left[a(t)(u(t) - y(t)) \right],$$

where (for the switching time $T = 0.5$ s)

$$a(t) = \begin{cases} 1.44 & \text{for } t < 0.5, \\ 2.1 & \text{for } t \geq 0.5, \end{cases} \qquad \alpha(t) = \begin{cases} -0.25 & \text{for } t < 0.5, \\ -0.5 & \text{for } t \geq 0.5. \end{cases}$$

The experimental results of fractional variable-order inertial system with order switching between -0.25 and -0.5, compared to their numerical results, are presented in Fig. 6.16.

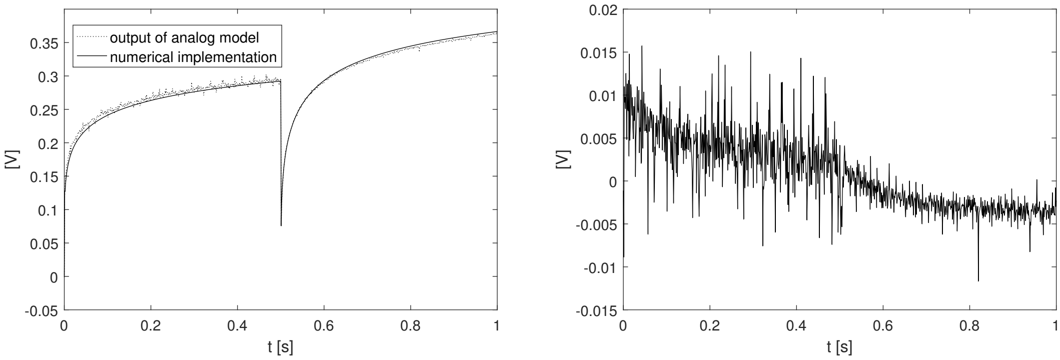


Fig. 6.15. Results of analog and numerical implementation of the $\mathcal{E}$-type inertial system with switching order from -0.25 to -0.5 (*left*) and their modeling error (*right*).

6.6.3 Results of the $\mathcal{E}$-type inertial systems with initial conditions

Realization of the $\mathcal{E}$-type fractional variable-order are shown in Fig. 6.13. To gather data for $\mathcal{E}$-type variable-order inertial system with initial conditions, the domino-ladder structure of fractional order impedance is firstly loaded to the initial value of voltage and then the scheme presented in 6.13 is running with $u(t) = 0$. Until switching time the S_1 switch is connected to terminal marked as a, and S_2 switch is connected to terminal b. After switching time switches change their positions. The initial value of voltage for all measurements is equal to 0.5 V and the sample time equals to 0.001 s.

Example 6.19. [29] Inertial system with initial conditions for order switching from -0.25 to -0.5.

In this case the scheme presented in Fig. 6.6 takes the following structure: Z_1, Z_2 are half-order impedances and resistors $R = 16\text{k}\Omega$. To have $\mathcal{E}$-type variable-order inertial system it is necessary to identify the parameters' of integral systems. The parameters' identification of quarter and half order integrals are shown in Fig. 6.10 and Fig 6.11.

The identified variable-order inertial system with infinite initial condition has the following form:

$$y(t) = {}^{\mathcal{E}}_{-\infty}\mathrm{D}_t^{\alpha(t)} \left[a(t)(u(t) - y(t))\right],$$

where (for the switching time $T = 0.4$ s)

$$a(t) = \begin{cases} 1.43 & \text{for } t < 0.4, \\ 2.1 & \text{for } t \geq 0.4, \end{cases} \qquad \alpha(t) = \begin{cases} -0.25 & \text{for } t < 0.4, \\ -0.5 & \text{for } t \geq 0.4. \end{cases}$$

The experimental results of fractional variable-order inertial system with initial condition for switching order between -0.25 and -0.5 at time $T = 0.4$s compared to their numerical results are presented in Fig. 6.16.

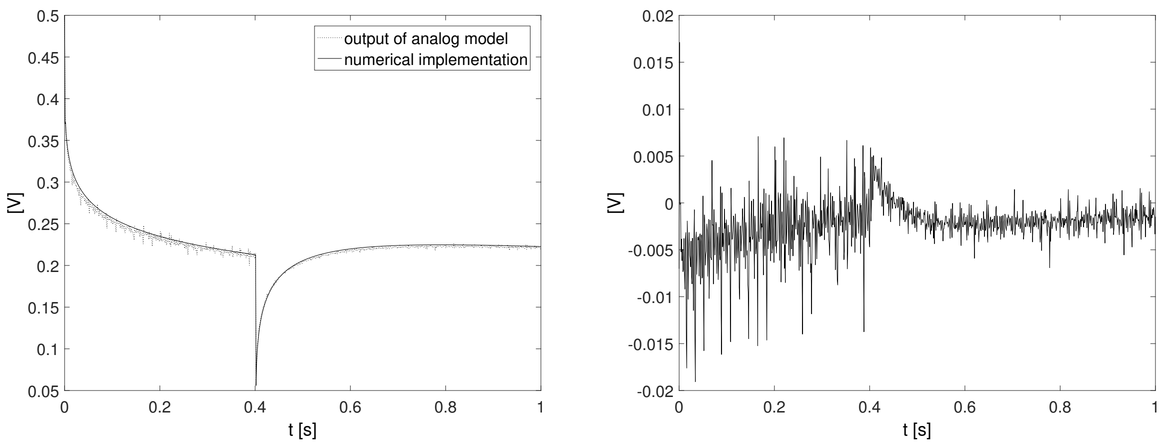


Fig. 6.16. Results of analog and numerical implementation of the $\mathcal{E}$-type inertial system with initial condition equals to 0.5V – switching from -0.25 to -0.5 (*left*) and their modeling error (*right*).

Example 6.20. [29] Inertial system with initial conditions for order switching from -0.5 to -1.

In this case the scheme presented in Fig. 6.6 takes the following structure: Z_1, Z_2 are half order impedances and resistors $R = 100\text{k}\Omega$. To have the variable-order inertial system it is necessary to make a parameters' identification of constant-order integrators. The identification results are obtained by numerical minimization of time responses square error with sampling time 0.001 s and input signal being $u(t) = 0.3 \cdot 1(t)$ V.

- When S_1 switch is connected to terminal marked as a and S_2 switch is connected to terminal b the identified system has the following form:

$$y(t) = 1.3 {}_0\mathrm{D}_t^{-0.5} u(t).$$

- When switch S_1 is connected to terminal b and S_2 is connected to terminal a the identified system has the following form:

$$y(t) = 1.75 {}_0\mathrm{D}_t^{-1} u(t).$$

The identification results of half- and first-order integral systems for input signal $u(t) = 0.3 \cdot 1(t)$ V are shown in Fig. 6.17 and 6.18 respectively. The identified $\mathcal{E}$-type variable-

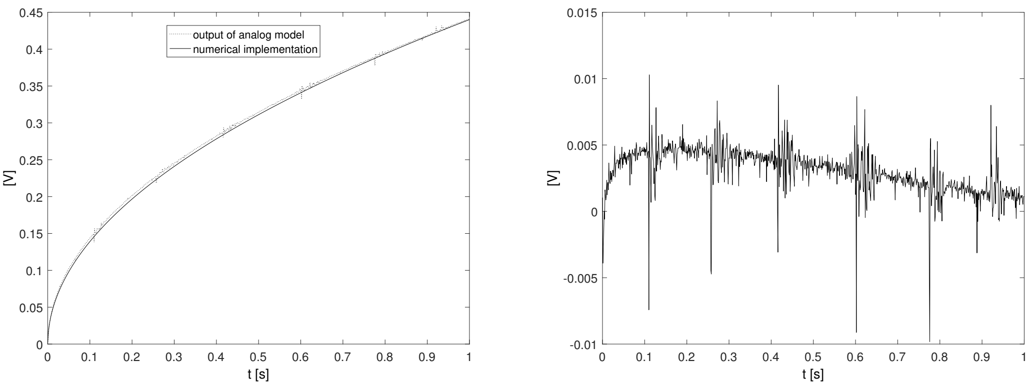


Fig. 6.17. Identification results of constant-order $\alpha = 0.5$ integrator (*left*) and their modeling error (*right*).

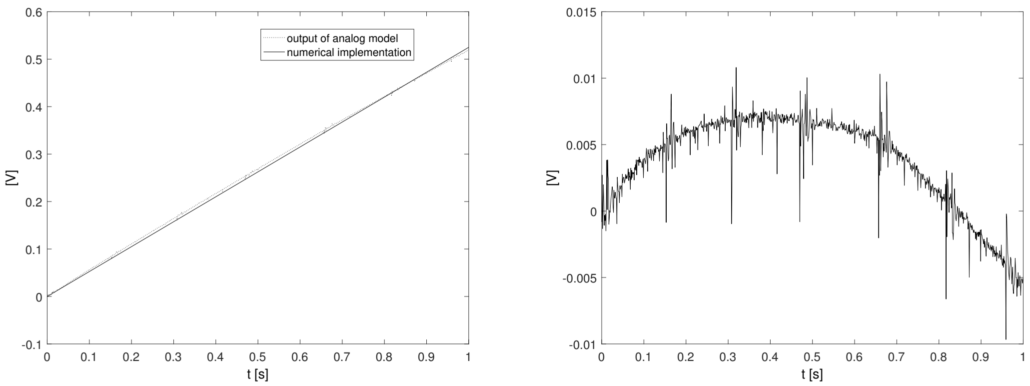


Fig. 6.18. Identification results of first-order integrator (*left*) and their modeling error (*right*).

order inertial system with infinite initial conditions has the following form:

$$y(t) = {}^{\mathcal{E}}_{-\infty}\mathrm{D}_t^{\alpha(t)} \left[a(t)(u(t) - y(t)) \right],$$

where (for the switching time $T = 0.3$ s.)

$$a(t) = \begin{cases} 1.3 & \text{for } t < 0.3, \\ 1.75 & \text{for } t \geq 0.3, \end{cases} \quad \alpha(t) = \begin{cases} -0.5 & \text{for } t < 0.3, \\ -1 & \text{for } t \geq 0.3. \end{cases}$$

The experimental results of fractional variable-order inertial system with initial condition for order switching between -0.5 and -1, compared to their numerical results, are presented in Fig. 6.19.

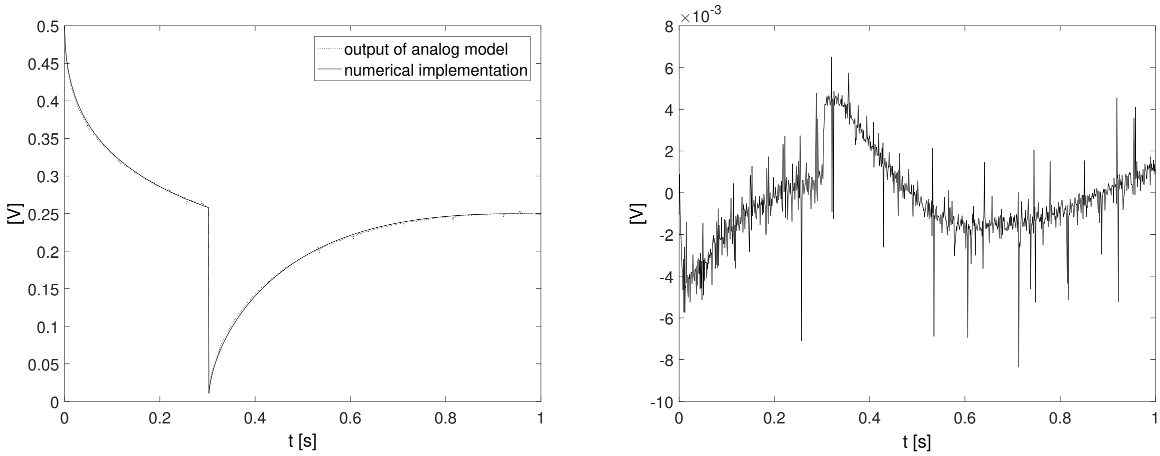


Fig. 6.19. Results of analog and numerical implementation of the $\mathcal{E}$-type fractional variable-order inertial system with initial condition equals to 0.5V – switching from -0.5 to -1 (*left*) and their modeling error (*right*).

6.7 Summary

The chapter consists of two parts. In the first part the theoretical background of $\mathcal{E}$-type fractional variable-order definition is introduced. It was proved that the $\mathcal{E}$-type fractional variable-order definition can be interpreted as the output-additive scheme. The numerical approach for that switching scheme was deeply investigated. Next, the method for considering the initial conditions in the form of finite and infinite length into $\mathcal{E}$-type fractional variable-order difference and into $\mathcal{E}$-type variable-order derivative was proposed. At the end of this part the composition properties of output-additive switching scheme with fractional constant-order differ-integral from the left-hand side and right-hand side were proved as well. It was shown, that the composition property holds, but only in one direction, it is not commutative and depends on the sequence of composition. Then, the theoretical considerations were experimentally validated in the second part of the chapter. The experimentally obtained data for $\mathcal{E}$-type: integral, inertial and inertial with initial conditions systems were compared to their numerical implementations and show high accuracy of presented method.

Chapter 7

Conclusions

The thesis is addressed to analog modeling of fractional variable-order dynamical systems. This work presents the particular types of fractional variable-order operators and their switching order structures. The idea of switching schemes gives an overview onto variable-order definitions and can be treated fairy as their interpretations. Moreover, presented switching order schemes can be used to design and create the fractional variable-order dynamical systems.

In the thesis, the identity between matrix forms of classical and recursive Grünwald-Letnikov operators was proved. Further, the $\mathcal{A}$-,$\mathcal{B}$-,$\mathcal{D}$- and $\mathcal{E}$-type of fractional variable-order operators were experimentally validated according to their switching order schemes. It was shown that based on the $\mathcal{A}$-,$\mathcal{B}$-,$\mathcal{D}$- and $\mathcal{E}$-type integrators, the more complicated variable-order dynamical systems can be realized. Next, the alternative multi-switching analog model of the $\mathcal{D}$-type definition was created. The major advantage of such alternative form, constructed directly based on half order impedance, is possibility to switch the order of integrator between 0.5 and 1 in any desire time. Considering the initial value of voltage occurring at domino ladder structures of fractional order impedance, the scheme of involving the initial conditions into $\mathcal{E}$-type derivative in the form finite and infinite time was proved and experimentally validated for the rest recursive operators (Grünwald-Letnikov, $\mathcal{D}$-type operator). It should be stressed out, that the initial value at the beginning of domino ladder structure has a big influence on the experimental results. Presented approach of initial conditions in the form of infinite time for recursive operators is related to the case, when all capacitors in the half and quarter order structures are loaded to the same value of voltage. The comparison of experimental and numerical data for all fractional variable-order models show high accuracy of proposed methods.

As was turned out, among the variable-order derivatives introducing in this thesis, the duality property can be applied. It means, that the composition of two particular types of variable-order operators with opposite sign value of orders gives an original function. Thus, in general, the order composition properties between the same types of variable-order operators does not holds. It was a main motivation to prove the order composition properties for the $\mathcal{E}$-type variable-order operator with fractional constant-order differ-integral in the Grünwald-Letnikov form. In this case, the side of composition has a crucial significance and the commutative law does not hold. So, the composition of the $\mathcal{E}$-type operator with constant-order definition from left-hand side holds, and for opposite case–does not.

As an overall conclusion for theoretical and experimental results, the final message of this work can be reminded: the particular types of fractional variable-order operators are equivalent to switching order structures, which can be used to analog modeling of fractional variable-order dynamical systems.

Using the variable-order operators and their analog models creates some problems. The first inconvenience is a growing up switching structure with every desire order. As long as order varying is representing by additional fractional constant-order derivative block (being the composition of current and desire order) at switching scheme, its analog model becomes more complicated with every desire order. Moreover, sometimes to achieve the desire order, the additional block refers to pure fractional constant-order derivative. It is a really difficult to do under experimental conditions. Additionally, author tried to validate experimentally the duality properties for variable-order operators. However, it was a crucial problem to obtain strictly the same parameters of particular analog models and what was already mentioned, in this case when one model refers to integrator, the second one becomes a derivative. Also, a small discrepancy between parameter values of particular analog models can be observed in experimental parts of the thesis, where despite of using the same structure and nominal values of passive elements, sometimes their identified parameters were slightly different. The inaccuracy of fractional constant-order impedances has an influence on experimental results as well.

This work contains the succinctly description of variable-order operators but it still does not exhaust the subject of variable-order calculus. Therefore, some open problems can be highlighted. The fractional variable-order definitions appearing in the thesis are in the form of Grünwald-Letnikov. However, the straight-forward definitions ($\mathcal{A}$-type,$\mathcal{B}$-type) posses their equivalent integral forms (see [26, 104]) against to recursive variable-order operators. So, finding the integral forms for the recursive fractional variable-order operators can be treated as a next natural step.

Another pending issue is to figure out the parallel structures equivalent to recursive variable-order operators. This may cause a situation, when a single fractional constant-order derivative block in parallel switching-order scheme will represent a desire (final) value of order. Thus, it will eliminate the problematic supplementary order block, occurring in series switching-order schemes.

The existence of variable-order dynamical models requires new efficient control design methods, which will recognize the varying nature of the model (plant). It is a next approaching issue which is worth to be taken into considerations.

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